

# ANALYTIC GEOMETRY

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## PUBLISHERS' NOTE

AMONG the papers of the late John Wesley Young there was found an unfinished manuscript on Analytic Geometry. As Professor Young was one of America's leading mathematicians, it seemed fitting that the manuscript be completed and published as a final tribute to his contribution to the field of Mathematics. At the time of his death, Professor Young was Editor-in-Chief of the Houghton Mifflin Company Mathematical Series. We therefore asked two mathematicians, who were close friends of Professor Young, to complete the manuscript. These men are Professor Tomlinson Fort of Lehigh University, a man well known for his research work in Mathematics and his ability as a teacher, and Dr. Frank Morgan, formerly on the faculty of Dartmouth College, a man who in the past collaborated with Professor Young in writing several books and original articles.

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## PREFACE

THIS text is designed to meet the requirements of the course in freshman Analytic Geometry as given in an Arts College or Engineering School. As many Arts Colleges require in addition to Analytic Geometry approximately a month's work in the calculus, a short chapter on this subject has been appended.

It will be found that the proofs are rigorous, and exceptional cases have not been passed over. The exercises, both numerical and analytical, are numerous and well graded. Approximately the last one third of the exercises in each group are problems to be assigned to the better students. In the discussion of parametric equations only curves which a student will encounter in his Elementary Calculus and Mechanics have been included.

Today many teachers prefer to use determinants where possible and the solution of examples by determinants has been included. However, if one wishes to omit determinants, the continuity of the book is not broken. Diameters, poles and polars have been treated under Loci and not as separate topics.

In answer to many requests the subject matter has been carefully restricted to those topics which most colleges include in an elementary course, thus relieving the instructor of the unpleasant task and necessity of selecting the material and omitting large sections of the text.

Special care has been taken to use type that is pleasing to the eye and easily read by artificial light.

The authors wish to thank Professor R. Beatley of Harvard, Professors B. H. Brown and C. H. Forsyth of Dartmouth, and Professor R. Johnson of Brooklyn College, for their many helpful suggestions and criticisms.

Typographical errors are bound to appear in the first edition of any mathematical text and the authors would appreciate learning about any such errors that may appear in this text. Please communicate with Dr. F. M. Morgan, Hanover, New Hampshire.

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## CHAPTER I

### INTRODUCTION

**1. Analytic geometry**, which is the study of geometry by means of algebraic methods, was presented in the first systematic way by the French mathematician René Descartes (1637) in his *La Géométrie*. This correlation between algebra and geometry marks one of the greatest advances in the field of mathematics. It has enabled mathematicians to give simple proofs for many geometric theorems which, until then, were difficult to prove, and, in addition, to make possible the discovery of many new theorems.

Before proceeding directly to the subject matter of the course, it is advisable to review a few of the topics of Elementary Mathematics which we shall have occasion to use.

#### **2. The general quadratic equation is**

$$ax^2 + bx + c = 0.$$

The two roots are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The sum of the roots is

$$x_1 + x_2 = -\frac{b}{a}.$$

The product of the roots is

$$x_1 x_2 = \frac{c}{a}.$$

The character of the roots depends on the quantity under the radical sign, namely,  $b^2 - 4ac$ , which is called the **discriminant**.

If  $b^2 - 4ac > 0$  the roots are real and unequal;

$b^2 - 4ac = 0$  the roots are real and equal;

$b^2 - 4ac < 0$  the roots are imaginary.

### Exercises

Without solving, find the nature, the sum, and the product of the roots of the following equations.

1.  $4x^2 - 7x + 3 = 0.$

2.  $4x^2 + 4x + 1 = 0.$

3.  $2x^2 + 7x - 11 = 0.$

4.  $6x^2 - 3x + 7 = 0.$

5.  $14x^2 - 2x - 1 = 0.$

6.  $16x^2 - 8x + 1 = 0.$

Determine  $k$  so that the roots of the following equations are equal.

7.  $x^2 + kx + 9 = 0.$

8.  $x^2 + (k+1)x + 16 = 0.$

9.  $kx^2 - 2(k+3)x + k + 7 = 0.$

10.  $(k+1)x^2 - (7k-1)x + 8k+1 = 0.$

11.  $(2k-1)x^2 - (k+1)x + (k-4) = 0.$

12.  $(2k-1)x^2 - 6kx + 4k+5 = 0.$

13. Determine  $k$  so that the roots of  $y^2 = 4x$  and  $x = 2y + k$  will be equal. (Note: Eliminate  $x$  and place  $b^2 - 4ac = 0$ .)

14. Determine  $k$  so that the roots of  $x^2 + y^2 = 25$ ,  $3x + 4y = k$  will be equal.
15. Determine  $k$  so that the roots of  $x^2 + y^2 = r^2$ ,  $y = mx + k$  will be equal.
16. Determine  $k$  so that the roots of  $y^2 = 4px$  and  $y = mx + k$  will be equal.

**3. Determinants.** The expression  $\frac{a_1 b_1}{a_2 b_2}$  which stands for  $a_1 b_2 - a_2 b_1$  is called a determinant of **order two**.

The expression 
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which stands for

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

or  $a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3$  is called a determinant of **order three**.

Similarly, the expression

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

which stands for

$$a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

is called a determinant of **order four**.

The same values of  $x$  and  $y$  will satisfy the equations

$$a_1x + b_1y + c_1 = 0,$$

$$a_2x + b_2y + c_2 = 0,$$

$$a_3x + b_3y + c_3 = 0,$$

when and only when

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

The result of eliminating  $x$ ,  $y$ , and  $z$  from three homogeneous equations

$$a_1x + b_1y + c_1z = 0,$$

$$a_2x + b_2y + c_2z = 0,$$

$$a_3x + b_3y + c_3z = 0,$$

is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

This determinant is called the **eliminant**.

### Exercises

Evaluate:

$$1. \begin{vmatrix} 2 & 4 \\ 7 & -3 \end{vmatrix}.$$

$$2. \begin{vmatrix} a & b \\ p & q \end{vmatrix}$$

$$3. \begin{vmatrix} 4 & 6 & 1 \\ 2 & 3 & 1 \\ 3 & -2 & 1 \end{vmatrix}$$

$$4. \begin{vmatrix} 4 & 8 & 1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \end{vmatrix}$$

<b>5.</b> $\begin{array}{cccc} 6 & 1 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{array}$	<b>6.</b> $\begin{array}{cccc} 4 & 8 & 2 & 7 \\ 3 & 2 & 2 & 2 \\ 4 & 2 & 4 & 5 \\ 5 & 6 & 3 & 3 \end{array}$
--	--

7. Eliminate  $x$  and  $y$  from the equations

$$4x + y + 2 = 0,$$

$$3x + y - k = 0,$$

$$2x - y + 3k = 0.$$

8. Eliminate  $x$  and  $y$  from the equations

$$2x - 2y + k = 0,$$

$$x = y - 7,$$

$$3x = 2y + k.$$

9. Eliminate  $A$ ,  $B$ , and  $C$  from the equations

$$Ax + By + C = 0,$$

$$Ax_1 + By_1 + C = 0,$$

$$Ax_2 + By_2 + C = 0.$$

10. Eliminate  $A$ ,  $B$ ,  $C$  from the equations

$$(x^2 + y^2) + Ax + By + C = 0,$$

$$(x_1^2 + y_1^2) + Ax_1 + By_1 + C = 0,$$

$$(x_2^2 + y_2^2) + Ax_2 + By_2 + C = 0,$$

$$(x_3^2 + y_3^2) + Ax_3 + By_3 + C = 0.$$

[*Hint:* The equations are homogeneous in 1,  $A$ ,  $B$ , and  $C$ .]

# INTRODUCTION

## 4. The fundamental trigonometric functions are

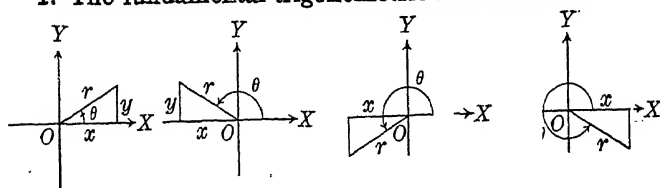


FIG. 1

$$\sin \theta = \frac{y}{r}, \quad \csc \theta = \frac{r}{y} \quad (y \neq 0),$$

$$\cos \theta = \frac{x}{r}, \quad \sec \theta = \frac{r}{x} \quad (x \neq 0),$$

$$\tan \theta = \frac{y}{x} \quad (x \neq 0), \quad \cot \theta = \frac{x}{y} \quad (y \neq 0).$$

The signs of the trigonometric functions in the four quadrants are given in the following table.

quadrant	1	2	3	4
sine cosecant	+	+	-	-
cosine secant	+	-	-	+
tangent cotangent	+	-	+	-

Each function of  $-\theta$ ,  $180^\circ \pm \theta$ ,  $360^\circ \pm \theta$  is equal

in absolute value (but not always in sign) to the same function of  $\theta$ .

Each function of  $90^\circ \pm \theta$ ,  $270^\circ \pm \theta$  is equal in absolute value (but not always in sign) to the corresponding co-function of  $\theta$ .

Some important trigonometric formulas are:

$$\sin^2 \theta + \cos^2 \theta = 1.$$

$$1 + \tan^2 \theta = \sec^2 \theta.$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}.$$

$$\sin (\theta_1 \pm \theta_2) = \sin \theta_1 \cos \theta_2 \pm \cos \theta_1 \sin \theta_2.$$

$$\cos (\theta_1 \pm \theta_2) = \cos \theta_1 \cos \theta_2 \mp \sin \theta_1 \sin \theta_2.$$

$$\tan (\theta_1 \pm \theta_2) = \frac{\tan \theta_1 \pm \tan \theta_2}{1 \mp \tan \theta_1 \tan \theta_2}.$$

$$\sin 2 \theta = 2 \sin \theta \cos \theta.$$

$$\cos 2 \theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta.$$

$$\tan 2 \theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

$$\tan \frac{\theta}{2} = \pm \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}.$$

It is well to remember the values given in the following table.

	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$
sine	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
cosine	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
tangent	0	$1/\sqrt{3}$	1	$\sqrt{3}$	no value

### Exercises

1. Find the sine, cosine, and tangent of  $0^\circ$ ;  $30^\circ$ ;  $45^\circ$ ;  $60^\circ$ ;  $90^\circ$ ;  $120^\circ$ ;  $135^\circ$ ;  $150^\circ$ ;  $180^\circ$ ;  $210^\circ$ ;  $225^\circ$ ;  $240^\circ$ ;  $270^\circ$ ;  $300^\circ$ ;  $315^\circ$ ;  $330^\circ$ ;  $360^\circ$ .

2. If  $\tan \theta = \frac{\sqrt{3}}{4}$ , and  $\sin \theta$  is positive, find the value of

$$\sin \frac{\theta}{2}; \cos \frac{\theta}{2}.$$

3. If  $\tan \theta = -\frac{\sqrt{3}}{12}$  and  $\sin \theta$  is positive, find the value of

$$\sin \frac{\theta}{2}; \cos \frac{\theta}{2}.$$

4. If  $\sin \alpha = \frac{3}{5}$  and  $\cos \alpha$  is positive while  $\cos \beta = -\frac{5}{13}$  and  $\sin \beta$  is positive, find the value of  $\tan (\alpha + \beta)$ ;  $\tan (\alpha - \beta)$ .

5. If  $\sin \alpha = \frac{4}{5}$  and  $\cos \alpha$  is negative, find the value of  $\sin 2\alpha$ ;

$$\cos 2\alpha; \tan 2\alpha; \sin \frac{\alpha}{2}; \cos \frac{\alpha}{2}; \tan \frac{\alpha}{2}.$$

6. If  $\tan 2\theta = \frac{x}{3}$  and  $2\theta$  is in the first quadrant, find  $\sin 2\theta$ ,  $\cos 2\theta$ ,  $\sin \theta$ ,  $\cos \theta$ .

7. If  $\theta_1 = 90^\circ + \theta_2$ , prove  $\tan \theta_1 = -\tan \theta_2$ .

8. Solve for  $\tan \theta$ , the equation  
 $4 \sin^2 \theta - 5 \sin \theta \cos \theta + \cos^2 \theta = 0$ .

[Hint: Divide by  $\cos^2 \theta$ ].

**5. Directed line-segments.** If a point moves in a straight line from  $A$  to  $B$ , we say it generates the segment  $AB$ , while if it moves in a straight line from  $B$  to  $A$  we say it generates the segment  $BA$ . The numerical measures of  $AB$  and  $BA$  are equal, but the directions of the segments are opposite. Therefore we use signed numbers to represent the segments. Thus, if  $AB = 5$  units,  $BA = -5$  units. It should be noted that in general  $AB = -BA$ .

Often we wish merely the magnitude or length of a directed segment  $AB$  in which case we write  $|AB|$ , read "The absolute value of  $AB$ ." For example, if  $AB = -5$ , then  $|AB| = |-5| = 5$ . If  $A$ ,  $B$ ,  $C$  are three points situated in any order on a straight line, then

$$(1) \qquad AB + BC = AC.$$

For, starting from  $A$  and moving to  $B$  and then to  $C$  is evidently the same as starting at  $A$  and moving directly to  $C$ . The segment  $AC$  is called the **sum of the segments  $AB$  and  $BC$** .

## Exercises

1. If  $A, B, C$  are three points on a directed line, find the value of:

a)  $AB + BC + CA$ .

b)  $BA + AC + CB$ .

If  $A, B, C, D$  are four points on a directed line find the value of:

2.  $AB + BC + CD$ .

3.  $AB - DC + BC$ .

4.  $BA + CB + DC$ .

5.  $|AB| + |BC| + |CA|$ .

6.  $AB + BC + CA$ .

7.  $|-AB| + |BC| + |-CD|$ .

8. If  $AB = 7$ ,  $BD = -9$ ,  $CD = 1$ , what is the algebraic value of  $AD$ ?  $CB$ ?  $AC$ ?

9. If  $AD = 5$ ,  $CB = -3$ ,  $DC = 7$ , what is the algebraic value of  $AB$ ?  $BD$ ?  $CA$ ?

10. If  $AC = 7$ ,  $CD = -3$ ,  $BD = -4$ , what is the algebraic value of  $DA$ ?  $AB$ ?  $BC$ ?

**6. Projection of a broken line.** If the perpendicular from a point  $P$  to a line  $l$  meets  $l$  in  $Q$ , the point  $Q$  is called the projection of  $P$  on  $l$ . If  $P$  is on  $l$ ,  $P$  is its own projection.

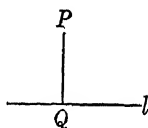


FIG. 2

Suppose  $P_1P_2$  to be a directed segment and  $l$  any arbitrarily chosen line. Let  $Q_1$  and  $Q_2$  be the projections of  $P_1$  and  $P_2$  respectively on  $l$ . Then we define the projection of  $P_1P_2$  on  $l$  to be the directed segment  $Q_1Q_2$  or the signed number which represents  $Q_1Q_2$ . Symbolically we write

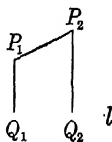


FIG. 3

(2)  $\text{Proj}_l P_1P_2 = Q_1Q_2$ .

Since  $Q_1Q_2 = -Q_2Q_1$  it follows that

$$(3) \quad \text{Proj}_l P_1P_2 = -\text{Proj}_l P_2P_1.$$

If there is no danger of confusion, the subscript showing the line on which we are projecting, is usually omitted.

The following theorems concerning projection will be found in most trigonometry texts:

**Theorem 1.** If  $P_1, P_2, P_3$  are any three points in a plane and  $l$  is any directed line in the plane, the algebraic sum of the projections of the segments  $P_1P_2$  and  $P_2P_3$  on  $l$  is equal to the projection of the segment  $P_1P_3$  on  $l$ .

**Theorem 2.** If  $A$  and  $B$  are any two points on a directed line  $p$ , and  $q$  is any directed line in the same plane with  $p$ , then we have both in magnitude and sign

$$(4) \quad \text{Proj}_q AB = AB \cos (qp),$$

where  $(qp)$  represents an angle through which  $q$  may be rotated in order to make its direction coincide with the direction of  $p$ .

### Exercises

1. If segment  $P_1P_2$  is parallel to line  $l$ , what can you say about the length of  $\text{Proj}_l P_1P_2$ ?
2. If segment  $P_1P_2$  or its prolongation is perpendicular to line  $l$ , what is the value of  $\text{Proj}_l P_1P_2$ ?
3. If  $P_1P_2 = 10$ ,  $\theta = 30^\circ$  and  $l$  is directed to the left, find  $\text{Proj}_l P_1P_2$ .

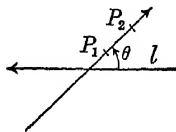


FIG. 4

4. If  $|OP| = 10$ ,  $\theta = 30^\circ$ , find  $|OM|$ ;  
 $|ON|$ .

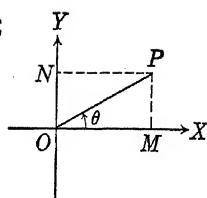


FIG. 5

5. If  $|OP| = 20$ ,  $\theta = 150^\circ$ , find  
 $|OM|$ ;  $|ON|$ .

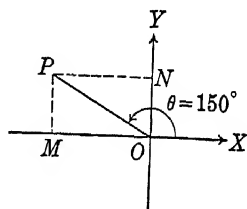


FIG. 6

## CHAPTER II

### THE POINT

**7. Coordinates on a line.** If on a directed line we choose a point  $O$  and a unit of length, every other point  $P$  on the line determines uniquely a directed segment  $OP$ , the algebraic value of which is also determined.

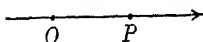


FIG. 7

This algebraic value is known as the **coordinate** of the point  $P$ . The point  $O$  is called the **origin**.

It is customary to choose the positive direction on a horizontal line to be the direction from left to right.

In establishing a system of coordinates on a line, the positive direction, the origin  $O$ , and the unit of length can be chosen arbitrarily, but when these have been selected, the coordinate  $x$  of every point  $P$  on the line is completely determined. Conversely, to every real value of  $x$  corresponds one and only one point. The system of coordinates is completely established by choosing the origin for which  $x = 0$  and the unit point for which  $x = +1$ .

If one is given a system of coordinates on a line with points  $P_1$  and  $P_2$  on the line having coordinates  $x_1$  and  $x_2$  respectively, i.e.,  $OP_1 = x_1$ ,  $OP_2 = x_2$ , then  $P_1 P_2 = x_2 - x_1$ . For,  $P_1 P_2 = P_1 O + OP_2 = -OP_1 + OP_2 = -x_1 + x_2 = x_2 - x_1$ .

1. Locate on a horizontal line, on which the origin and the unit point have been marked, the points whose coordinates are the following numbers, 2, 5, -3, -7,  $\frac{2}{3}$ ,  $\sqrt{2}$ ,  $\sqrt{3}$ .

2. Each of the following pairs of numbers give the coordinates of a pair of points on a horizontal line. Find the value of the segment from the first point to the second and interpret the sign.

$$a) 3, -4. \quad c) -\frac{1}{2}, \frac{1}{2}. \quad e) x+7, x+11.$$

$$b) -3, -4. \quad d) x, x+2. \quad f) -7, 0.$$

3. Each of the following sets of three numbers represents the coordinates of three points  $A, B, C$  on a line. Verify for each set that  $AB + BC = AC$  by finding the algebraic values of the segments  $AB, BC$ , and  $AC$ .

$$a) 1, 2, 4. \quad c) -1, 6, -5.$$

$$b) 2, 5, 3. \quad d) 12, 5, -7.$$

4. Given a system of coordinates on a line and points  $P_1$  and  $P_2$  on the line with coordinates  $x_1$  and  $x_2$  respectively. Prove that the coordinate of the mid-point of the segment  $P_1P_2$  is  $\frac{1}{2}(x_1 + x_2)$ .

5. Determine the coordinate of the mid-point of each of the segments situated on a horizontal line, the coordinates of whose end points are:

$$a) 2, 4. \quad c) 6, -3.$$

$$b) -1, 7. \quad d) -7, -4.$$

**8. Coordinates in a plane.** To locate a point in a plane we make use of two mutually perpendicular lines,  $OX$  and  $OY$ , on each of which a scale has been established, such that the origin on each scale is at the point of intersection  $O$  of the two lines and

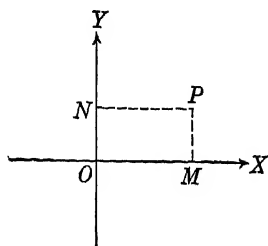


FIG. 8

such that the positive direction on  $OY$  is obtained from the positive direction of  $OX$ , by rotating  $OX$  through a right angle in a counterclockwise direction. The units of length on the two scales need not be equal but are usually taken so.

If, now,  $P$  is any point in the plane and lines are drawn through  $P$  parallel to  $OY$  and  $OX$ , these lines will meet  $OX$  and  $OY$  in points  $M$  and  $N$  respectively. The coordinate  $x = OM$  of the point  $M$  in the coordinate system on  $OX$  is called the  **$x$ -coordinate** or **abscissa** of  $P$ ; the coordinate  $y = ON = MP$  of the point  $N$  in the coordinate system on  $OY$  is called the  **$y$ -coordinate** or **ordinate** of  $P$ . The two numbers  $(x, y)$  are called the **coordinates** of  $P$  referred to the axes  $OX$  and  $OY$ . The latter are called the **coordinate axes**,  $OX$  being the  **$x$ -axis** and  $OY$  the  **$y$ -axis**.

Any point  $P$  in the plane determines uniquely two coordinates  $x$  and  $y$ . Conversely, to every pair of numbers,  $x$  and  $y$ , there corresponds a point  $P$  whose coordinates are  $(x, y)$ .

The coordinate axes  $OX$  and  $OY$  divide the plane into four regions called **quadrants** numbered as in the figure. These quadrants are identical to those used in the study of trigonometry. Points on the axes are exceptional and do not lie in any quadrant. It will be noted that any point in the first quadrant is characterized by the fact that both of its coordinates are positive, while the coordinates of any

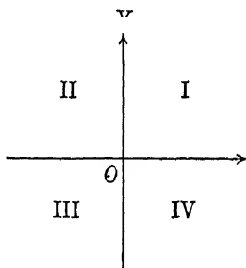


FIG. 9

point in the third quadrant are both negative. A point in the second quadrant has its abscissa negative and its ordinate positive, while a point in the fourth quadrant has its abscissa positive and its ordinate negative.

### Exercises

1. Plot the points  $(1, 2)$ ,  $(3, 5)$ ,  $(-2, 4)$ ,  $(2, -4)$ ,  $(-1, -3)$ .
2. What is the ordinate of every point on the  $x$ -axis?
3. What is the abscissa of every point on the  $y$ -axis?
4. What are the coordinates of the origin?
5. In what quadrants are the abscissas positive? negative? ordinates positive? negative?
6. In what quadrant is a point if its abscissa is negative and its ordinate positive? If its abscissa is positive and its ordinate negative?
7. Draw the quadrilateral whose vertices are  $(-4, -1)$ ,  $(3, -1)$ ,  $(3, 2)$ ,  $(-4, 2)$ . Prove the quadrilateral has its opposite sides equal.
8. The origin is the middle point of a line one of whose extremities is  $(3, 4)$ . Find the coordinates of the other extremity.
9. Find the coordinates of the point midway between the origin and the point  $(6, 0)$ .
10. The rectangle  $ABCD$  has its base  $AB$  horizontal. If the  $x$ -axis lies along  $AB$  and the  $y$ -axis along  $AD$ , find the coordinates of  $A, B, C, D$  if  $AB = 10$ ,  $AD = 3$ .

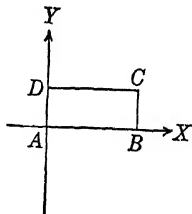


FIG. 10

11. Find the coordinates of the vertices of a square whose side is 10 if the origin is at the center of the square and the axes are parallel to the sides.
12. The base of an equilateral triangle with side 10, coincides with the  $x$ -axis; the center of the base is the origin. Find the coordinates of the vertices. Two solutions.
13. Plot the points  $P_1(3, 2)$ ,  $P_2(6, 6)$  and compute the distance  $P_1P_2$ .  
[Hint: Draw horizontal and vertical lines through  $P_1$  and  $P_2$  respectively, meeting at  $M$ . Find lengths  $P_1M$ ,  $MP_2$  and use the theorem of Pythagoras.]
14. Plot the points  $P_1(0, 4)$ ,  $P_2(6, 8)$  and compute the distance  $P_1P_2$ .
15. Plot the points  $P_1(-4, 3)$ ,  $P_2(6, -1)$  and compute the distance  $P_1P_2$ .

**9. Projections of a segment  $P_1P_2$  on the axes.** Let  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  be any two points in the plane.

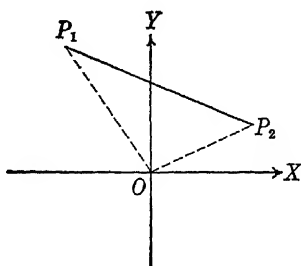


FIG. 11

Connect  $O$  to  $P_1$  and  $P_2$ . Then applying Theorem 1, § 6,  
 $\text{Proj}_x P_1P_2 = \text{Proj}_x P_1O + \text{Proj}_x OP_2 = -\text{Proj}_x OP_1 + \text{Proj}_x OP_2$ .

But,  $\text{Proj}_x OP_1 = x_1$ ,  $\text{Proj}_x OP_2 = x_2$ .

$$(1) \quad \therefore \text{Proj}_x P_1P_2 = x_2 - x_1.$$

Likewise the projection of  $P_1P_2$  on any line parallel to the  $x$ -axis and similarly directed is  $x_2 - x_1$ .

In the same manner we can prove

$$(2) \quad \text{Proj}_y P_1P_2 = y_2 - y_1$$

and the projection of  $P_1P_2$  on any line parallel to the  $y$ -axis and similarly directed is  $y_2 - y_1$ .

*Example.* Given  $P_1(3, 4)$ ,  $P_2(-4, -7)$ . Find  $\text{Proj}_x P_1P_2$ ;  $\text{Proj}_x P_2P_1$ ;  $\text{Proj}_y P_1P_2$ ;  $\text{Proj}_y P_2P_1$ .

$$\text{Solution: } \text{Proj}_x P_1P_2 = -4 - 3 = -7.$$

$$\text{Proj}_x P_2P_1 = 3 - (-4) = 7.$$

$$\text{Proj}_y P_1P_2 = -7 - 4 = -11.$$

$$\text{Proj}_y P_2P_1 = 4 - (-7) = 11.$$

**10. The distance between two points.** Given two points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  which determine the

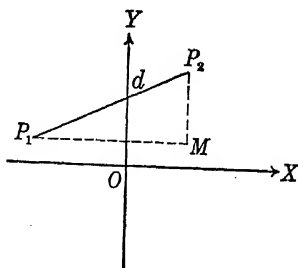


FIG. 12

segment  $P_1P_2$ . To find the length  $d$  of this segment, draw a line through  $P_1$  parallel to the  $x$ -axis and a

line through  $P_2$  parallel to the  $y$ -axis and let these lines meet at  $M$ .

By the Pythagorean Theorem

$$(3) \quad (P_1P_2)^2 = (P_1M)^2 + (MP_2)^2.$$

But from the last article,

$$(4) \quad P_1M = x_2 - x_1, \quad MP_2 = y_2 - y_1.$$

In (3)  $P_1M$  and  $MP_2$  represent merely magnitudes while in (4) they represent magnitudes and directions. However, if we square the quantities in (4) the results are the squares of the magnitudes and hence we are justified in substituting these values in (3).

$$\therefore \overline{P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

$$(5) \quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

*Example.* Find the distance between the points  $(-2, 5)$  and  $(7, -3)$ .

Solution:

$$d = \sqrt{(-2 - 7)^2 + (5 + 3)^2} = \sqrt{81 + 64} = \sqrt{145}.$$

### Exercises

- Find the distance between each of the following pairs of points:

$$a) (2, 4), (1, 6).$$

$$b) (-3, 1), (4, -2).$$

$$c) (-3, -5), (-6, 2).$$

- Find the distance between:

$$a) (a, b), (-a, -b).$$

$$b) (a + b, a + c), (a + c, b + c).$$

3. Find the length of the sides of the triangle whose vertices are  $A(5, 1)$ ,  $B(0, 10)$ ,  $C(7, 8)$ . Is the triangle isosceles? right?
4. Is the triangle  $A(-2, 1)$ ,  $B(2, 4)$ ,  $C(3, 1)$  equilateral? isosceles? right?
5. Show that the opposite sides of the quadrilateral  $(-5, -3)$ ,  $(1, -11)$ ,  $(7, -6)$ ,  $(1, 2)$  are equal and hence that the figure is a parallelogram.
6. Prove that the figure  $(3, 2)$ ,  $(0, 5)$ ,  $(-3, 2)$ ,  $(0, -1)$  is a rhombus. Show that the diagonals are equal and hence that the figure is a square.
7. Find a point on the  $x$ -axis equidistant from  $(2, 4)$  and  $(6, 8)$ .
8. Find a point on the  $y$ -axis equidistant from  $(-2, 4)$  and  $(6, 8)$ .
9. Show that the points  $(5, 4)$ ,  $(4, -3)$ ,  $(-2, 5)$  are equidistant from  $(1, 1)$ . What is the center and radius of the circle on which the first three points are situated?
10. The segment  $AB$  is 13 units in length. The coordinates of  $A$  are  $(4, -8)$  and the ordinate of  $B$  is  $-3$ . Find the abscissa of  $B$ . Two solutions.
11. Prove that the points  $(6, 4)$ ,  $(-1, 2)$ ,  $(3, -2)$ ,  $(2, 8)$  are the vertices of a parallelogram.
12. What kind of a triangle has vertices at  $(6, -2)$ ,  $(1, -2)$ ,  $(-2, 2)$ ?
13. Express by an equation the fact that the square of the distance from the point  $(x, y)$  to the point  $(2, -1)$  is 5.
14. Express by an equation the fact that the point  $(x, y)$  is equidistant from  $(2, 4)$  and  $(6, 8)$ .
15. Express by an equation the fact that the distance from  $(x, y)$  to  $(2, 3)$  is twice the distance from  $(x, y)$  to  $(3, 4)$ .

16. Express by an equation the fact that the sum of the distances from  $(x, y)$  to  $(-2, 0)$  and  $(2, 0)$  is 6.
17. Express by an equation the fact that the difference of the distances from  $(x, y)$  to  $(2, 0)$  and  $(-2, 0)$  is 1.

**11. Mid-point of segment.** Let  $P(x, y)$  be the mid-point of the segment connecting  $P_1(x_1, y_1)$  to  $P_2(x_2, y_2)$ . Let the projections of  $P_1$ ,  $P$ , and  $P_2$  on the  $x$ -axis be  $M_1$ ,  $M$ , and  $M_2$ . Since the directed segment  $P_1P$  equals the directed segment  $PP_2$  it follows that

$$\text{Proj}_x P_1P = \text{Proj}_x PP_2,$$

or 
$$x - x_1 = x_2 - x.$$

$$\therefore x = \frac{x_1 + x_2}{2}.$$

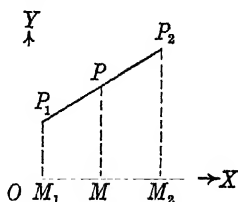


FIG. 13

Similarly by projecting on the  $y$ -axis,

$$y = \frac{y_1 + y_2}{2}.$$

Therefore, the coordinates of the mid-point are

$$(6) \quad \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

*Example.* Find the mid-point of the segment joining

$$(4, -8) \text{ to } (-3, 4).$$

$$\text{Solution: } x = \frac{4 - 3}{2} = \frac{1}{2}; \quad y = \frac{-8 + 4}{2} = \frac{-4}{2} = -2.$$

12. The point dividing  $P_1P_2$  in a given ratio. If three points  $P_1, P, P_2$  are collinear and  $P$  lies between  $P_1$  and  $P_2$ , it is said to divide the segment  $P_1P_2$  **internally** in the ratio  $\frac{P_1P}{PP_2}$ , while if  $P$  lies without the segment it

is said to divide the segment **externally** in the ratio  $\frac{P_1P}{PP_2}$ .

Let  $P(x, y)$  be the point which divides the segment  $P_1(x_1, y_1)$  to  $P_2(x_2, y_2)$  in an arbitrary ratio  $\frac{r_1}{r_2}$ , i.e.,

$$\frac{P_1P}{PP_2} = \frac{r_1}{r_2}.$$

The projections of  $P_1P$  and  $PP_2$  on the  $x$ -axis are in the same ratio as  $\frac{r_1}{r_2}$ .

$$\therefore \frac{x - x_1}{x_2 - x} = \frac{r_1}{r_2}.$$

Solving for  $x$ , we have

$$x = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}.$$

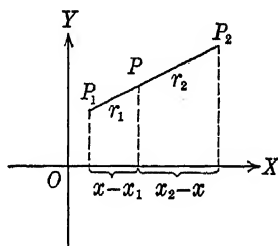


FIG. 14

$$\text{Similarly, } y = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}.$$

Hence the coordinates of the required point are

$$(7) \quad \left( \frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2} \right).$$

In applying (7) it must be remembered that  $r_1$  and  $r_2$  need not be the exact lengths  $P_1P$  and  $PP_2$  but any

numbers proportional to these lengths. Moreover,  $r_1$  is proportional to the segment nearest  $P_1$  and  $r_2$  to the segment nearest  $P_2$ .

If  $r_1 = r_2$ , formulas (7) reduce to those for the mid-point given in Art. 11.

If the point  $P$  is on the line  $P_1P_2$  extended, then  $P$  divides  $P_1P_2$  externally,  $P_1P$  and  $PP_2$  have opposite signs and the ratio  $\frac{r_1}{r_2}$  is negative.

*Example 1.* Find the coordinates of the point which divides the segment from  $(-6, 3)$  to  $(3, 9)$  in the ratio 1:2.

Solution: If  $P_1$  is  $(-6, 3)$ , then  $P_2$  is  $(3, 9)$  and  $r_1 = 1$  and  $r_2 = 2$ .

$$x = \frac{(1)(3) + (2)(-6)}{1 + 2} = \frac{-9}{3} = -3,$$

$$y = \frac{(1)(9) + (2)(3)}{1 + 2} = \frac{15}{3} = 5.$$

*Example 2.* Find the point which divides the line from  $(8, 5)$  to  $(-4, 7)$  externally in the ratio 3:2.

Solution:  $P_1$  is  $(8, 5)$ ,  $P_2$  is  $(-4, 7)$ . Either  $r_1$  or  $r_2$  must be negative. Let  $r_1 = 3$ , then  $r_2 = -2$ .

$$\text{Then } x = \frac{(3)(-4) + (-2)(8)}{3 - 2} = \frac{-12 - 16}{1} = -28,$$

$$y = \frac{(3)(7) + (-2)(5)}{3 - 2} = \frac{21 - 10}{1} = 11.$$

## Exercises

1. Find the coordinates of the mid-point of the segment joining:
  - a)  $(8, 4)$ ,  $(10, 8)$ .
  - b)  $(6, -4)$ ,  $(4, 8)$ .
  - c)  $(-3, -2)$ ,  $(4, 7)$ .
  - d)  $(-2, 7)$ ,  $(-5, 0)$ .
2. Find the coordinates of the mid-points of the sides of the triangle with vertices  $(2, -3)$ ,  $(-7, 4)$ ,  $(3, 6)$ .
3. Find the points of trisection of the segments in Ex. 1.
4. Find the coordinates of the point which divides the line from  $(8, 4)$  to  $(10, 8)$  internally in the ratio  $3:2$ .
5. Find the coordinates of the point which divides the segment from  $(-2, 3)$  to  $(4, 7)$  externally in the ratio  $3:7$ .
6. Find the coordinates of the point which divides the segment from  $(4, 7)$  to  $(-3, 2)$  externally in the ratio  $4:5$ .
7. In what ratio does the point  $(3, -2)$  divide the segment connecting  $(5, -4)$  to  $(-1, 2)$ ?
8. The mid-point of a line is  $(2, 3)$  and one extremity is  $(-4, 2)$ . Find the coordinates of the other extremity.
9. The segment  $AB$  is produced to  $P$  so that  $BP = 2 AB$  in length. If  $A$  is  $(2, 4)$  and  $B$  is  $(-3, 2)$ , find the coordinates of  $P$ .
10. Find the point of intersection of the medians of the triangle  $A(4, 7)$ ,  $B(3, 2)$ ,  $C(7, 5)$ .  
[Hint: Find  $M$  the mid-point of  $AB$ . Find  $P$  which divides  $CM$  in ratio  $2:1$ .]

11. Find the point of intersection of the medians of the triangle  $(3, 4)$ ,  $(6, 2)$ ,  $(10, 8)$ .
12. Find the point of intersection of the medians of the triangle  $(12, 6)$ ,  $(6, 8)$ ,  $(4, -10)$ .
13. The segment  $P_1(4, -9)$ ,  $P_2(-4, y_2)$  is divided by  $P(x, 7)$  so that  $P_1P:PP_2 = 4:5$ . Find  $x$  and  $y_2$ .
14. If  $(x-h)^2 + (y-k)^2 = 100$ , how far is it from the point  $(x, y)$  to the point  $(h, k)$ ?
15. The segment  $P_1(-4, y_1)$ ,  $P_2(x_2, 17)$  is bisected by  $P(1, 10)$ . Find  $x_2$  and  $y_1$ .
16. Three vertices of a parallelogram named in order are  $A(-4, 2)$ ,  $B(-2, -4)$ ,  $C(6, 3)$ . Find the fourth vertex  $D$ . How many parallelograms are there if the points are not taken in the order named?

**13. Angle of inclination and slope of line joining two points.** The smallest angle through which the positive half of the  $x$ -axis,  $OX$ , must be turned counter-clockwise in order to bring it parallel to the given line  $P_1P_2$  is called the **angle of inclination** of  $P_1P_2$ .

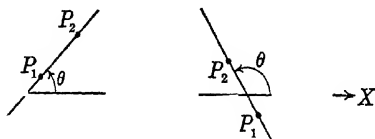


FIG. 15

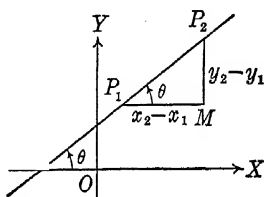
Any given line  $P_1P_2$  will either cut the  $x$ -axis or be parallel to it.

If line cuts the  $x$ -axis, the angle of inclination  $\theta$ , will always be less than  $180^\circ$ , while if  $P_1P_2$  is parallel to the  $x$ -axis the angle of inclination is either  $0^\circ$  or  $180^\circ$ .

In place of the angle of inclination it is often more convenient to use the tangent of the angle which is called the **slope** of the line and is designated by the letter  $m$ .

Thus,  $m = \tan \theta$ .

$$\begin{aligned} \text{Now } \tan \theta &= \frac{MP_2}{P_1M} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \end{aligned}$$



$$(8) \quad m = \frac{y_2 - y_1}{x_2 - x_1}.$$

FIG. 16

If we had taken the segment  $P_2P_1$  we should have obtained

$$m = \frac{y_1 - y_2}{x_1 - x_2} \text{ which is equal to (8), which demonstrates,}$$

that  $m$  is the same no matter which way the line is directed.

The student should make clear to himself the significance of the term "slope of a line." It is a number which measures the "steepness" of the line; the greater the numerical value of the slope, the "steeper" is the line. If the slope is a positive number, a point moves upward as it travels from left to right on the line; if the slope is a negative number, the point moves downward as it travels from left to right on the line.

If the slope is zero, the line is parallel to the  $x$ -axis and conversely.

Since  $\tan \theta$  is undefined when  $\theta = 90^\circ$ , a line perpendicular to the  $x$ -axis **has no slope**. Care should be

taken not to confuse the statement "has no slope" with the statement "the slope is zero."

If points  $P_1$  and  $P_2$  determine a line parallel to the  $y$ -axis, it should be noted that  $x_2 = x_1$  and formula (8) has no meaning.

*Example 1.* Construct the line through the point  $P(3, 4)$  with a slope equal to  $\frac{1}{2}$ .

*Solution:* Lay off a horizontal segment  $PM$  of two units to the right of  $P$  and from the end point  $M$  of this segment lay off a vertical segment  $MR$  upward of one unit. The line  $PR$  is the required line, since  $\tan \theta = \frac{1}{2}$ .

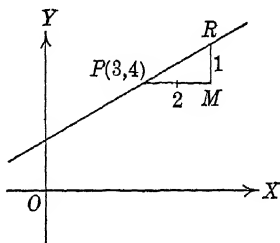


FIG. 17

*Example 2.* Construct the line through the point  $P(3, 4)$  whose slope is  $-\frac{1}{2}$ .

*Solution:* Lay off a horizontal segment  $PM$  of two units as before; lay off a vertical segment  $MR$  downward, equal to one unit.  $PR$  is the desired line, since

$$\tan \theta = \frac{MR}{PM} = -\frac{1}{2}.$$

If the segment  $PM$  had been laid off to the left two units and the segment  $MR$  upward one unit, the result would have been the same.

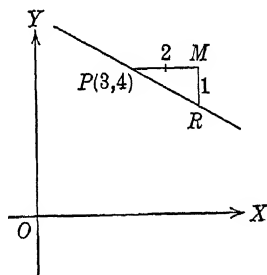


FIG. 18

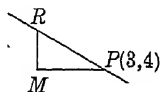


FIG. 19

## Exercises

Find the slope of the lines joining the following pairs of points:

1.  $(2, 4), (4, 2)$ .
2.  $(2, 1), (-3, 4)$ .
3.  $(-4, 7), (2, 4)$ .
4.  $(-1, 4), (6, -2)$ .
5.  $(-2, -3), (4, -3)$ .
6.  $(7, 1), (7, -4)$ .

Show that the following sets of points are collinear, *i.e.*, are on a straight line.

7.  $(0, 0), (2, 3), (4, 6)$ .
8.  $(0, 0), (1, -2), (-2, 4)$ .
9.  $(0, 0), (2, 1), (-4, -2)$ .
10.  $(1, 4), (2, 3), (4, 1)$ .
11.  $(0, 0), (-1, -2), (3, 6)$ .
12.  $(3, 1), (3, 4), (3, -2)$ .
13.  $(4, 7), (-3, 7), (2, 7)$ .

Determine  $h$  so that the three points in each of the following sets shall be collinear.

14.  $(1, -1), (3, 1), (h, 4)$ .
15.  $(2, 1), (4, 3), (0, h)$ .
16.  $(0, 2), (1, 1), (h, 3)$ .
17.  $(1, -1), (-2, 5), (2, h)$ .
18.  $(4, 6), (-3, 8), (h, h)$ .

Construct the line through the point  $P$  with the slope  $m$ , given.

19.  $P(4, 3), m = \frac{3}{4}$ .
20.  $P(-2, 2), m = 3$ .
21.  $P(-4, 3), m = -\frac{1}{3}$ .
22.  $P(4, 2), m = -5$ .
23.  $P(4, 2), m = 0$ .
24.  $P(-2, 3), m = -\frac{1}{2}$ .

**14. Slopes of parallel and perpendicular lines.** If two lines are parallel their angles of inclination are equal. Since  $\theta_1 = \theta_2$ ,  $\tan \theta_1 = \tan \theta_2$ , and  
 (9)  $m_1 = m_2$ .

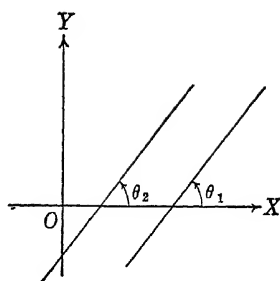


FIG. 20

Hence, if two lines are parallel, and have slopes, the slopes are equal. Conversely, if the slopes of two lines are equal, the lines are parallel. If  $\theta_1 = \theta_2 = 90^\circ$  the lines are parallel but have no slopes. Conversely, if two lines have no slopes they are parallel.

Suppose the angles of inclination of  $l_1$  and  $l_2$  are  $\theta_1$  and  $\theta_2$  respectively and  $l_2 \perp l_1$ .

$$\text{Now } \theta_2 = 90^\circ + \theta_1.$$

$$\therefore \tan \theta_2 = \tan (90^\circ + \theta_1).$$

$$= -\cot \theta_1$$

$$= -\frac{1}{\tan \theta_1}$$

$$= -\frac{1}{m_1}$$

$$(10) \text{ Hence } m_2 = -\frac{1}{m_1}.$$

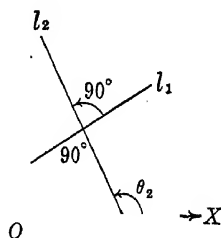


FIG. 21

That is to say, if two lines are perpendicular and have slopes, these slopes are negative reciprocals of each other.

Conversely, if the slopes of two lines are nega-

tive reciprocals of each other, the lines are perpendicular.

If a line has a slope 0, it is parallel to the  $x$ -axis. Any line perpendicular to this line is perpendicular to the  $x$ -axis and has no slope.

### Exercises

1. Find the slopes of the sides of the triangle with vertices  $(5, 7)$ ,  $(6, 2)$ ,  $(3, 5)$  and prove the triangle is a right triangle.
2. Prove that the line through  $(-2, -4)$ ,  $(3, 3)$  is parallel to the line through  $(1, -2)$ ,  $(6, 5)$ .
3. Prove that the line joining  $(6, 0)$  to  $(0, 4)$  is parallel to the line joining  $(3, 0)$  to  $(0, 2)$  and twice as long.
4. Prove that the opposite sides of the quadrilateral  $(2, 8)$ ,  $(6, 4)$ ,  $(3, -2)$ ,  $(-1, 2)$  are parallel.
5. Prove that the figure  $(3, 4)$ ,  $(2, 1)$ ,  $(1, 3)$ ,  $(4, 2)$  is a rectangle.
6. Prove that the line joining  $(4, 9)$  to  $(2, 5)$  is parallel to the line joining  $(5, 2)$  to  $(6, 4)$  but is perpendicular to the line joining  $(6, 2)$  to  $(8, 1)$ .
7. Prove that the figure formed by joining the mid-points of the sides of the quadrilateral  $(4, 6)$ ,  $(10, 8)$ ,  $(12, -6)$ ,  $(-2, -4)$  in order, is a parallelogram.
8. Express by an equation the fact that the slope of the line joining  $(x, y)$  to  $(4, 7)$  is 3.
9. Express by an equation the fact that the slope of the line joining  $(x, y)$  to  $(2, 3)$  is equal to the slope of the line joining  $(2, 3)$  to  $(4, 7)$ .

10. Express by an equation the fact that the slope of the line joining  $(x, y)$  to  $(2, 3)$  is equal to the slope of the line joining  $(x, y)$  to  $(4, 7)$ .
11. Three vertices of a parallelogram named in order are,  $A(-4, 2)$ ,  $B(-2, -4)$ ,  $C(6, 3)$ . Find the fourth vertex. [Hint: Call the fourth vertex  $(x, y)$ . Find an equation in  $x$  and  $y$  by placing slope of  $DA = \text{slope } CB$ ; find a second equation in  $x$  and  $y$  by placing slope of  $DC = \text{slope of } AB$ . Solve these equations for  $x$  and  $y$ .]
12. Three vertices of a parallelogram are  $A(-2, 4)$ ,  $B(4, 6)$ ,  $C(10, -2)$ . Find the fourth vertex. Three solutions.

**15. The proof of theorems of elementary geometry by means of analytic geometry.**

*Example 1.* Prove analytically that the diagonals of a rectangle are equal.

Solution: Suppose  $P_1P_2P_3P_4$  to be any rectangle. In order to use the methods of analytic geometry it is necessary to establish a coordinate system. Any pair of perpendicular lines could be used but the work is simplified greatly by a judicious choice. We shall take the  $x$ -axis along  $P_1P_2$  and the  $y$ -axis along  $P_1P_4$ . If the sides of the rectangle are  $a$  and  $b$ , the vertices are  $P_1(0, 0)$ ,  $P_2(a, 0)$ ,  $P_3(a, b)$ ,  $P_4(0, b)$ . Then

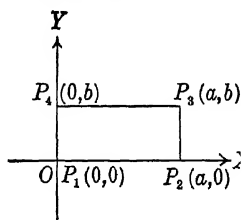


FIG. 22

$$P_1P_3 = \sqrt{a^2 + b^2}, \quad P_2P_4 = \sqrt{a^2 + b^2}. \quad \therefore P_1P_3 = P_2P_4.$$

*Example 2.* Prove analytically that if two medians

of a triangle are equal, the triangle is isosceles.

Solution: Let the triangle be  $P_1P_2P_3$ . Let the  $x$ -axis pass through  $P_1P_2$  and the  $y$ -axis through  $P_3$  perpendicular to  $P_1P_2$ . The coordinates\* of the vertices can be taken as  $P_1(2a, 0)$ ,  $P_2(2b, 0)$ ,  $P_3(0, 2c)$ .

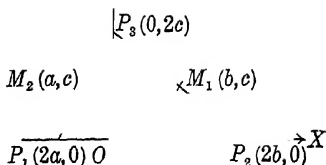


FIG. 23

Then mid-point  $M_1$  is  $(b, c)$  and mid-point  $M_2$  is  $(a, c)$ .

By hypothesis  $P_1M_1 = P_2M_2$ .

$$\therefore \sqrt{(b-2a)^2 + c^2} = \sqrt{(2b-a)^2 + c^2}.$$

Simplifying,  $a^2 = b^2$  or  $a = \pm b$ . But  $a$  cannot equal  $b$ . Why? If  $a = -b$ ,  $OP_1 = -OP_2$  or  $P_1O = OP_2$ , i.e., the altitude from  $P_3$  bisects the base and hence the triangle is isosceles.

*Example 3.* Prove analytically that the diagonals of a parallelogram bisect each other.

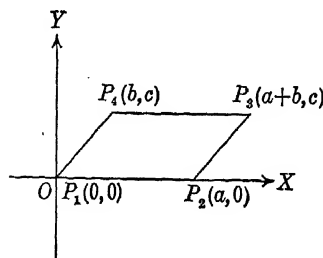


FIG. 24

\* We have used  $2a, 2b, 2c$ , in this case rather than  $a, b, c$ , so as to avoid fractions for the coordinates of the mid-points.

Solution: Let  $P_1P_2P_3P_4$  be any parallelogram. Let the  $x$ -axis pass through  $P_1P_2$  and  $y$ -axis through  $P_1$  perpendicular to  $P_1P_2$ . We can take the coordinates of  $P_1$  and  $P_2$  as  $(0, 0)$ , and  $(a, 0)$  respectively. Call the coordinates of  $P_4$ ,  $(b, c)$ . Since  $P_4P_3 = P_1P_2 = a$ , and since the distance of  $P_4$  from the  $y$ -axis is  $b$ , the abscissa of  $P_3$  is  $a + b$ . But  $P_4P_3$  is parallel to  $P_1P_2$  and hence the ordinate of  $P_3$  is  $c$ . Therefore, the coordinates of  $P_3$  are  $(a + b, c)$ .

The coordinates of the mid-point of both  $P_1P_3$  and  $P_2P_4$  are

$$\left( \frac{a + b}{2}, \frac{c}{2} \right)$$

and hence the diagonals bisect each other.

### Exercises

1. Prove analytically that the line joining the mid-points of two sides of a triangle is equal to half the third side and parallel to it.
2. Prove analytically that the line joining the mid-point of the hypotenuse of a right triangle to the vertex of the right angle is equal to half the hypotenuse.
3. Prove analytically that the line joining the mid-points of the non-parallel sides of a trapezoid is equal to half the sum of the parallel sides.
4. Prove analytically that the diagonals of a square are perpendicular to each other.
5. Prove analytically that the line joining the mid-points of two opposite sides of a parallelogram is parallel to the other two sides and equal to each of them.

6. Prove analytically that the lines joining the mid-points of the sides of a square taken in order form a square.
7. Prove analytically that the medians to the legs of an isosceles triangle are equal.
8. Prove analytically that every point in the perpendicular bisector of a line segment is equidistant from the ends of the segment.
9. Prove analytically that if the diagonals of a parallelogram are equal, the figure is a rectangle.
10. Prove analytically that the lines joining the mid-points of the opposite sides of a quadrilateral bisect each other.
11. Prove analytically that the lines joining in order the mid-points of the sides of any quadrilateral form a parallelogram.
12. Prove analytically that in any quadrilateral the lines joining the mid-points of a pair of opposite sides and the mid-points of the diagonals bisect each other.
13. In the parallelogram  $ABCD$ ,  $P$  and  $Q$  trisect  $AC$ . Prove analytically that  $BP = DQ$ .
14. Prove analytically that if each half of the diagonals of a parallelogram is bisected and the mid-points are connected in order, that the figure formed is a parallelogram.
15. In the parallelogram  $ABCD$ ,  $E$  and  $F$  bisect  $AD$  and  $BC$  respectively. Prove analytically that  $BFEA$  is a parallelogram.
16. In the parallelogram  $P_1P_2P_3P_4$ ,  $M$  is the mid-point of  $P_1P_2$ . Prove analytically that  $P_3M$  and diagonal  $P_2P_4$  trisect each other.
17. Prove analytically that if the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, the triangle is a right triangle.

18. If  $D$  is the mid-point of the side  $BC$  of triangle  $ABC$ , prove analytically

$$\overline{AB}^2 + \overline{AC}^2 = 2 \overline{AD}^2 + 2 \overline{BD}^2.$$

19. Prove analytically that in any quadrilateral the sum of the squares of the sides is equal to the sum of the squares of the diagonals plus four times the square of the distance between the mid-points of the diagonals.
20. Prove analytically that the lines joining the vertices of a triangle to the mid-points of the opposite sides meet in a point and trisect each other.
21. In triangle  $ABC$ , the medians  $AD$  and  $BE$  meet at  $O$ ; prove analytically that the length of the line joining the mid-points  $R$  and  $S$  of  $AO$  and  $BO$  is equal to the length of  $DE$ .

## CHAPTER III

### EQUATION OF A LOCUS LOCUS OF AN EQUATION

**16. Equation of a locus.** By the locus\* of a point is meant the totality of points which satisfy a given geometric condition.

If the position of the point is determined by means of coordinates, the geometric condition governing its position is then expressed by an equation which is satisfied by the coordinates of all points on the locus and is not satisfied by the coordinates of points not on the locus. This equation is called the **equation of the locus** of the point  $P$ .

*Example 1.* Find the equation of the locus of a point 3 units to the right of the  $y$ -axis.

Solution: The required equation is  $x = 3$ , for the equation is satisfied by the coordinates of every point 3 units to the right of the  $y$ -axis and is not satisfied by the coordinates of any other point.

*Example 2.* Find the equation of the locus of points equidistant from the points whose coordinates are  $(2, 2)$  and  $(6, 4)$ .

Solution: Let the given points be  $A$  and  $B$  and let  $P(x, y)$  be the point whose locus we desire. The given geometric condition is

$$(1) \quad PA = PB.$$

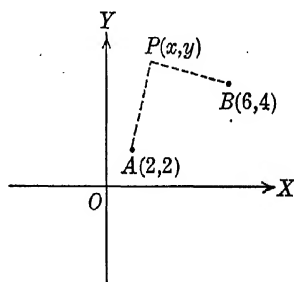


Fig. 25

\* In this book the word curve means locus.

Then

$$\sqrt{(x-2)^2 + (y-2)^2} = \sqrt{(x-6)^2 + (y-4)^2},$$

$$\begin{aligned} x^2 - 4x + 4 + y^2 - 4y + 4 &= \\ x^2 - 12x + 36 + y^2 - 8y + 16, \end{aligned}$$

$$\text{or} \quad -4x + 4 - 4y + 4 = -12x + 36 - 8y + 16,$$

$$8x + 4y - 44 = 0,$$

$$(2) \quad 2x + y - 11 = 0.$$

That is, if the coordinates of the point  $P$  satisfy (1), they satisfy (2). Conversely, if  $(x, y)$  satisfy (2), by retracing our steps, we see they satisfy (1).

Hence the equation of the locus of  $P$  is

$$2x + y - 11 = 0.$$

*Example 3.* Find the equation of the locus of a point such that the sum of its distances from  $(3, 0)$  and  $(-3, 0)$  is 10.

Solution: Let  $F$  and  $F'$  be respectively the points  $(3, 0)$ ,  $(-3, 0)$ , and  $P(x, y)$  any point on the required locus.

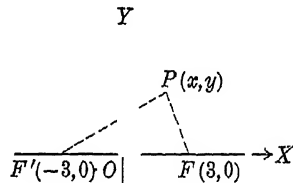


FIG. 26

$$(3) \text{ Then } PF + PF' = 10.$$

$$\sqrt{(x-3)^2 + y^2} + \sqrt{(x+3)^2 + y^2} = 10,$$

$$\text{or} \quad \sqrt{(x-3)^2 + y^2} = 10 - \sqrt{(x+3)^2 + y^2}.$$

Squaring both members,

$$x^2 - 6x + 9 + y^2 =$$

$$100 - 20\sqrt{(x+3)^2 + y^2} + x^2 + 6x + 9 + y^2.$$

Collecting terms,

$$20 \sqrt{(x+3)^2 + y^2} = 12x + 100,$$

or 
$$5 \sqrt{(x+3)^2 + y^2} = 3x + 25.$$

Squaring,

$$25x^2 + 150x + 225 + 25y^2 = 9x^2 + 150x + 625.$$

Simplifying,

$$(4) \qquad 16x^2 + 25y^2 = 400.$$

That is, if the coordinates of a point  $P$  satisfy (3) they satisfy (4). Conversely, if the coordinates of a point satisfy (4), by retracing our steps, and remembering that when we extract the square root that there are two signs, we have the four equations

$$\pm \sqrt{(x-3)^2 + y^2} \pm \sqrt{(x+3)^2 + y^2} = 10.$$

These four equations can be denoted as follows:

$$\begin{array}{ll} a) & + + \\ c) & - + \\ b) & + - \\ d) & - - \end{array}$$

We wish to show that  $a)$  is the only one of the four equations which is true. Equations  $b)$  and  $c)$  state that the difference of the distances  $PF$  and  $PF'$  is equal to 10 and hence greater than  $FF'$  which is 6. This is absurd, for the difference of two sides of a triangle is less than the third side. Equation  $d)$  is false for the left hand member is always negative and hence can never equal the positive quantity 10. Hence, if the coordinates of a point satisfy (4), they satisfy  $a)$  which

is the same as (3). Therefore the equation of the locus is

$$16x^2 + 25y^2 = 400.$$

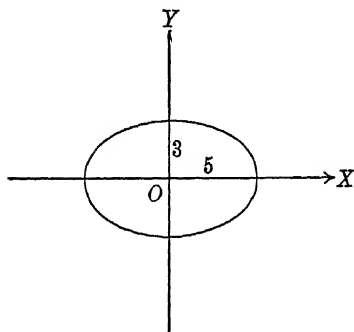


FIG. 26a.

### Exercises

1. What is the equation of the locus of a point which moves 4 units to the left of the  $y$ -axis?
2. What is the equation of the locus of a point which moves 3 units below the  $x$ -axis?
3. Find the equation of the locus of points in the first and third quadrants, three times as far from the  $x$ -axis as from the  $y$ -axis.
4. Find the equation of the locus of points in quadrants two and four, five times as far from the  $y$ -axis as from the  $x$ -axis.

Find the equation of the locus of a point:

5. Equidistant from  $(4, 6)$  and  $(8, 10)$ .
6. Equidistant from  $(2a, 2b)$  and  $(4b, 4a)$ .
7. If its distance from  $(-4, 6)$  is 5.
8. If its distance from  $(a, b)$  is  $a$ .

9. If its distance from  $(4, 0)$  is equal to its distance from the  $y$ -axis.
10. If its distance from  $(0, -2)$  is equal to its distance from the  $x$ -axis.
11. If the difference of the squares of its distances from  $(2, 0)$  and  $(6, 0)$  is 4.
12. If the sum of the squares of its distances from  $(2, 0)$  and  $(6, 0)$  is 16.
13. If the ratio of its distance from  $(2, 3)$  to its distance to  $(4, 6)$  is 2.
14. If the ratio of its distance from  $(4, 5)$  to its distance to  $(3, 6)$  is 5.
15. If it is twice as far from  $(3, 0)$  as from  $(-2, 4)$ .
16. If it is three times as far from  $(4, 0)$  as from  $(0, 4)$ .
17. If the sum of its distances from  $(2, 0)$  and  $(-2, 0)$  is equal to 10.
18. If the difference of its distances from  $(2, 0)$  and  $(-2, 0)$  is 3.
19. If the sum of its distances from  $(3, 0)$  and  $(-3, 0)$  is equal to 10.
20. If the difference of its distances from  $(3, 0)$  and  $(-3, 0)$  is equal to 5.

**17. The locus of an equation.** We have seen that if a system of coordinate axes is set up, then to every pair of real numbers  $(x, y)$  there corresponds a point in the plane. If  $x$  and  $y$  are variables connected by an equation, then this equation will in general be satisfied by an infinite number of pairs of values of  $x$  and  $y$ , each pair of values being the coordinates of a point. These

points are not, however, scattered indiscriminately over the plane, but usually lie on a curve whose form will vary according to the equation under consideration. This curve is called the *locus* of the given equation. An important problem of Analytic Geometry is to determine the locus of a given equation.

**18. Plotting the locus of an equation.** If we assign a series of values to one coordinate, say  $x$ , we can then determine the corresponding value of the other coordinate  $y$ . By this process we can determine the coordinates of a series of points which are situated on the required graph or curve. If we plot a sufficient number of these points and draw a smooth curve through them, we have an approximation of the required curve. By choosing a sufficiently large number of points close to each other, this approximation will vary but slightly from the required curve. The following examples will illustrate the method.

*Example 1.* Plot the locus of  $y = x + 1$ .

*Solution:* Assigning values to  $x$  and computing the corresponding values of  $y$ , we construct the table

$x$	- 3	- 2	- 1	0	1	2	3	4
$y$	- 2	- 1	0	1	2	3	4	5

Plotting these points we see they appear to lie on a straight line, but we cannot conclude that this is the case. However, in § 33 we shall prove that they are actually on a straight line.

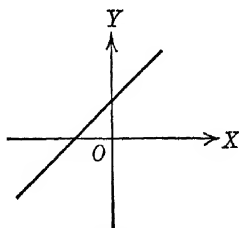


FIG. 27

*Example 2.* Plot the locus of  $x^2 - y^2 = 4$ .

*Solution:* Solving for  $y$ , we have  $y = \pm$

It is evident that  $y$  is imaginary for values of  $x$  greater than  $-2$  and less than  $2$ . Making a tabular representation we have

$x$	$\pm 2$	$\pm 3$	$\pm 4$	$\pm 5$	$\pm 6$
$y$	0	$\pm 2.2$	3.5	4.6	5.6

Plotting these points and drawing a smooth curve through them, we see they appear to determine a two-branched curve.

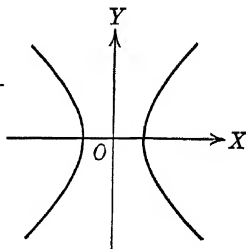


FIG. 28

### Exercises

Plot the loci of the following equations:

1.  $x = 4$ .
2.  $x = 0$ .
3.  $5x = 0$ .
4.  $y = 3$ .
5.  $y = -3$ .
6.  $y = 0$ .
7.  $my = 0$ .
8.  $x - y = 0$ .
9.  $x + y = 0$ .
10.  $2x + y = 0$ .
11.  $3x + y = 4$ .
12.  $2x - y = 5$ .
13.  $2x - y = 0$ .
14.  $2x + 3y = 12$ .
15.  $y = 2x^2$ .
16.  $y = -3x^2$ .
17.  $y = 3x^2 - 4$ .
18.  $x = y^2$ .
19.  $x = 3y^2 - 4$ .
20.  $x^2 + y^2 = 4$ .
21.  $x^2 + y^2 = 25$ .
22.  $x^2 + y^2 = 9$ .
23.  $4x^2 + y^2 = 36$ .
24.  $9x^2 + y^2 = 36$ .
25.  $x^2 - y^2 = 9$ .

26.  $x^2 - y^2 = 25$ .

27.  $x^2 - y^2 = 36$ .

28.  $xy = 4$ .

29.  $xy = -2$ .

30.  $y = -x^2$ .

31.  $xy + y = 4$ .

32.  $xy - 2x = 8$ .

33.  $y = x^3$ .

**19. Discussion of an equation.** The process of determining a locus by plotting points is a satisfactory method in simple examples, but when the equation is complicated, the method is usually too laborious and tedious. It may often be shortened by considering the following four properties of the curve:

- a) Intercepts.
- b) Symmetry.
- c) The range of values of the variables.
- d) Asymptotes.

**20. Intercepts.** The  $x$ -intercepts of a curve are the abscissas of the points where the curve meets the  $x$ -axis, while the  $y$ -intercepts are the ordinates of the points where the curve meets the  $y$ -axis. Hence to find the intercepts of a locus whose equation is given, we proceed as follows:

- I.  $x$ -intercepts. Place  $y = 0$  and solve the resulting equation for  $x$ .
- II.  $y$ -intercepts. Place  $x = 0$  and solve the resulting equation for  $y$ .

*Example.* Find the intercepts of  $4x^2 + 9y^2 = 36$ .

Solution: Placing  $y = 0$ , we have  $4x^2 = 36$  or  $x = \pm 3$ .

$\therefore$   $x$ -intercepts are  $\pm 3$ .

Placing  $x = 0$  we have  $9y^2 = 36$  or  $y = \pm 2$ .

$\therefore$   $y$ -intercepts are  $\pm 2$ .

### Exercises

Find the intercepts of:

1.  $x^2 + y^2 = 9$ .
2.  $y^2 = 4x + 4$ .
3.  $x^2 - y^2 = 4$ .
4.  $y^2 = 2x^2 - 8$ .
5.  $x + y = 12$ .
6.  $y - x = 0$ .
7.  $xy = 4$ .
8.  $y^3 - x^2 = 27$ .
9.  $x^3 + y^3 = 27$ .
10.  $x^2 = 4y$ .
11.  $x^2 = 8y + 2x$ .
12.  $y^2 - xy - 4 = 0$ .

**21. Symmetry.** Two points  $A$  and  $A'$  are said to be symmetric with respect to a line  $l$ , if  $l$  is the perpendicular bisector of the segment  $AA'$ .

A curve is symmetric with respect to a line  $l$ , if the curve is made up of pairs of points symmetric with respect to  $l$ .

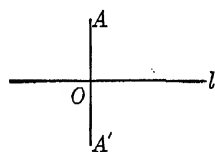


FIG. 29

Two points  $A$  and  $A'$  are said to be symmetric to a third point  $O$ , if the point  $O$  is the mid-point of the segment  $AA'$ . A curve is symmetric with respect to a given point if it is made up of pairs of points which are symmetric with respect to the given point.

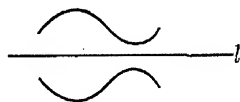


FIG. 30

- I. If replacing  $y$  by  $-y$  does not change the equation of a locus, the locus is symmetric with respect to the  $x$ -axis.

For, if  $x$  and  $y$  are a pair of numbers satisfying the given equation, then by hypothesis  $x$  and  $-y$  also satisfy. Geometrically this means the symmetric points  $(x, y)$  and  $(x, -y)$  are on the curve. Therefore, every point on the curve has its symmetric point also on the curve.

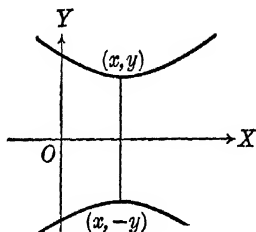


FIG. 31

- II. If replacing  $x$  by  $-x$  does not change the equation of a locus, the locus is symmetric with respect to the  $y$ -axis.
- III. If replacing  $x$  by  $-x$  and  $y$  by  $-y$  does not change the equation of a locus, the locus is symmetric with respect to the origin.
- The proofs of rules II and III are left to the student.

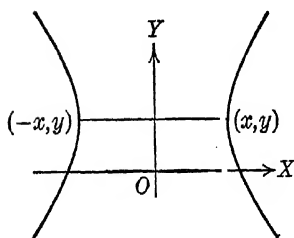


FIG. 32

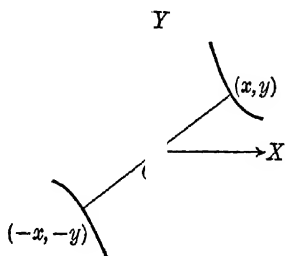


FIG. 33

*Example.* Test for symmetry the locus  $4x^2 - y^2 = 4$ .

*Solution:*

I. Replacing  $y$  by  $-y$  we have

$$4x^2 - (-y)^2 = 4 \text{ or } 4x^2 - y^2 = 4.$$

Hence the curve is symmetric with respect to the  $x$ -axis.

II. Replacing  $x$  by  $-x$ , we have

$$\begin{aligned} 4(-x)^2 - y^2 &= 4 \\ \text{or } 4x^2 - y^2 &= 4. \end{aligned}$$

Hence the curve is symmetric with respect to the  $y$ -axis.

III. Replacing  $x$  by  $-x$ ,  $y$  by  $-y$ , we have

$$\begin{aligned} 4(-x)^2 - (-y)^2 &= 4 \\ \text{or } 4x^2 - y^2 &= 4. \end{aligned}$$

Hence the curve is symmetric with respect to the origin.

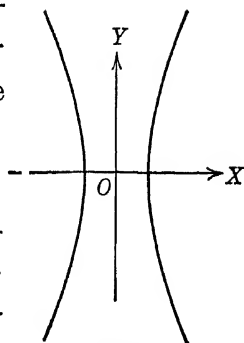


FIG. 34

### Exercises

Test for symmetry the loci of the following equations:

1.  $x = y$ .

2.  $y^2 = 4x$ .

3.  $x^2 + y^2 = 9$ .

4.  $x^2 + 4y^2 = 36$ .

5.  $x^2 - 4y^2 = 36$ .

6.  $y = x^3$ .

7.  $y^2 = x^3$ .

8.  $y^2 - 2x + 1 = 0$ .

9.  $x^2 + 3y - 2 = 0$ .

10.  $xy = 5$ .

11.  $xy = -5$ .

12.  $xy + y^2 - x^2 = 6$ .

13.  $y^2 - 2xy = 4$ .

14.  $y^3 = x^2 + 2x - 1$ .

15.  $x^2y = 5$ .

16.  $y = \frac{x e^x + 1}{2 e^x - 1}$

**22. The range of values of the variable.** There are often intervals in which there is no curve. These intervals can often be determined:

- 1) By solving the equation for  $y$ , and determining between what values of  $x$  the values of  $y$  are imaginary.
- 2) By solving for  $x$  and determining for what values of  $y$  the values of  $x$  are imaginary.

*Example 1.* Determine if there are any excluded regions for the curve whose equation is  $4x^2 - y^2 = 36$ . Sketch the curve.

Solution: Solving for  $y$ , we have

$$y = \pm 2 \sqrt{(x-3)(x+3)}.$$

In order for  $y$  to be real,  $(x-3)(x+3)$  must either be zero or positive. Therefore, there is no curve between  $x = 3$  and  $x = -3$ .

Solving for  $x$ , we have

$$x = \pm \frac{1}{2} \sqrt{36 + y^2}.$$

Since  $36 + y^2$  is positive for every real value of  $y$ , there are no excluded values of  $y$ . A tabular representation is

$x$	$\pm 3$	$\pm 4$	$\pm 5$	$\pm 6$
$y$	0	$\pm 2\sqrt{7}$	$\pm 8$	$\pm 6\sqrt{3}$

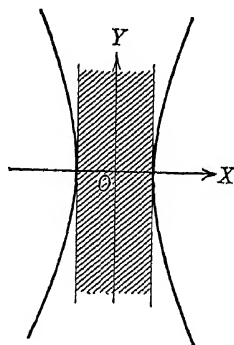


FIG. 35

Noting that the curve is symmetric with respect to the  $x$ -axis,  $y$ -axis, and the origin, the graph is readily constructed.

*Example 2.* Determine if there are any excluded regions for the curve whose equation is

$$y^2 + x^2 - 4x + 3 = 0. \quad \text{Sketch the curve.}$$

Solution: Solving for  $y$ , we have

$$y = \pm \sqrt{(x-1)(3-x)}.$$

In order for  $y$  to be real,  $(x-1)(3-x)$  must either be zero or positive. Therefore the curve is situated only between  $x = 1$  and  $x = 3$ .

Solving for  $x$ , we have

$$x = 2 \pm \sqrt{(1-y)(1+y)}.$$

In order for  $x$  to be real,  $(1-y)(1+y)$  must either be zero or positive. Therefore the curve is situated only between  $y = 1$  and  $y = -1$ .

Making out a table of values

$x$	1	1.5	2	2.5	3
$y$	0	$\pm \sqrt{.75}$	$\pm 1$	$\pm \sqrt{.75}$	0

and noting that the curve is symmetric with respect to the  $x$ -axis the graph is readily constructed.

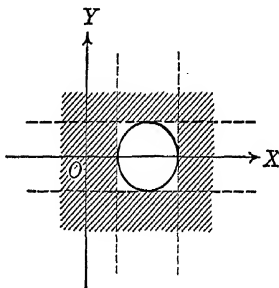


FIG. 36

### Exercises

Determine any excluded regions for the following loci and sketch the curves:

1.  $y^2 = 4x$ .

2.  $x^2 = y$ .

3.  $y^2 = -4x$ .

4.  $x^2 = -3y$ .

- |                       |                            |
|-----------------------|----------------------------|
| 5. $x^2 + y^2 = 16$ . | 6. $x^2 + 4y^2 = 36$ .     |
| 7. $x^2 - y^2 = 4$ .  | 8. $2x - y = 4$ .          |
| 9. $x^2 = 4y + 4$ .   | 10. $x^2 + y^2 + 2x = 0$ . |
| 11. $y^2 = x^3$ .     | 12. $y^2 = -x^3$ .         |

23. **Asymptotes.** If an open branch curve is thought of as generated by a point and if when the generating point recedes indefinitely along that branch the curve approaches coincidence with a fixed straight line, then the fixed line is called an **asymptote** of the curve.

We shall consider only asymptotes parallel to the co-ordinate axes. The following example will illustrate the method of determining asymptotes.

*Example.* Sketch the locus of the equation

$$y = \frac{8}{x^2 - 4}$$

**Solution:**

1. If  $x = 0$ ,  $y = -2$ . If  $y = 0$  there is no corresponding value of  $x$ . Hence there is a  $y$ -intercept of  $-2$  and no  $x$ -intercept.
2. If we replace  $x$  by  $-x$  the equation is left unchanged. Hence the curve is symmetric with respect to the  $y$ -axis. The curve is not symmetric with respect to the  $x$ -axis or origin. Why?
3. Factoring the denominator, we have

$$y = \frac{8}{x^2 - 4} = \frac{8}{(x - 2)(x + 2)}$$

There is a value of  $y$  corresponding to every value of  $x$ , except  $x = 2$  and  $x = -2$ . Suppose  $x$  is a little less than 2 and is allowed to increase and approach 2 as a limit;  $y$  then increases negatively without limit; that is, as a point moves indefinitely down along the curve, the curve approaches coincidence with the line  $x = 2$ .

If  $x$  is slightly greater than 2 and is allowed to decrease and approach 2 as a limit,  $y$  increases positively without limit; that is, as a point moves indefinitely far up along the curve, the curve approaches coincidence with the line  $x = 2$ .

The line  $x = 2$  is therefore an asymptote. From symmetry we see  $x = -2$  is also an asymptote.

Solving the equation for  $x$ , we have

$$x = \pm 2 \sqrt{\frac{2+y}{y}}$$

Values of  $y$  for which  $\frac{2+y}{y}$

is negative must be excluded. For the fraction to be negative, the numerator and denominator must have opposite signs. This is true in the interval

$$-2 < y < 0$$

and hence in this interval there is no curve.

Moreover, reasoning as we did for the vertical asymptotes, we find that the line with equation

$$y = 0$$

is an asymptote.

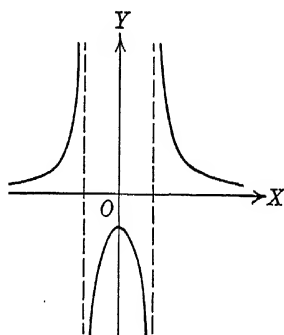


FIG. 37

**Rule.** To find vertical asymptotes, solve the equation for  $y$  in terms of  $x$  and factor the denominator. To every factor  $x - a$ , there corresponds a vertical asymptote whose equation is  $x = a$ .

To find horizontal asymptotes, solve the equation for  $x$  in terms of  $y$  and factor the denominator. To every factor  $y - b$ , there corresponds a horizontal asymptote whose equation is  $y = b$ .

### Exercises

Discuss and plot the loci of the following equations:

1.  $y = x.$

2.  $y^2 = x.$

3.  $y = x^2.$

4.  $xy = 4.$

5.  $x^2 + y^2 = 25.$

6.  $x^2 + 4y^2 = 36.$

7.  $x^2 - y^2 = 4.$

8.  $y^2 = x - 4.$

9.  $y^2 = 4 - x.$

10.  $y^2 - x^2 = 9.$

11.  $xy - 2y = 4.$

12.  $xy - 3y + 2 = 0.$

13.  $y^2(x - 2) = 4.$

14.  $y^2(x - 3) + 2 = 0.$

15.  $y(x - 1)(x - 2) = 4.$

16.  $x(y - 1)(y - 2) = 4.$

17.  $y^2 = x^2 - 1.$

18.  $y^2 = x^3.$

19.  $y^2 = (x - 1)(x - 2)(x - 3).$

20.  $y^2 + x - x^2y^2 = 0.$

21.  $y(x - 1)^2 = 4.$

22.  $(y - 1)^2 = (x - 3)^3.$

23.  $y - x + yx^2 = 0.$

24.  $y^2(1 + x^2) = 1.$

25.  $y(x - 2)(x - 3) = x + 3.$

26.  $y^2(x^2 - 1) = 1.$

**24. Condition that a point lie on curve.** Since the equation of a curve is an equation which connects the coordinates  $(x, y)$  of a variable point, it follows that **a point lies on a given curve if and only if its coordinates satisfy the equation of the curve.** For example, since  $x = 4, y = 3$  satisfies  $x^2 + y^2 = 25$ , the point  $(4, 3)$  lies on the curve represented by the given equation, while the point  $(2, 1)$  is not on the curve since its coordinates do not satisfy the given equation.

**25. Points of intersection of two loci.** If two loci, whose equations are given, meet in a point whose coordinates are  $(x_1, y_1)$  then  $x_1, y_1$  satisfies both equations. Conversely, if two equations are satisfied by  $x_1, y_1$  it follows that the point  $(x_1, y_1)$  must lie on both loci. Therefore, **to find the points of intersection of two loci, solve their equations simultaneously.**

If the equations have no real common solution, the curves do not intersect.

*Example.* Find the points of intersection of  $x^2 = 4y$  and  $y - 3x + 5 = 0$ .

Solution: From the second equation,  $y = 3x - 5$ . Substituting the value of  $y$  in the first equation, we have

$$x^2 = 4(3x - 5),$$

or  $x^2 - 12x + 20 = 0$ . Hence  $(x - 10)(x - 2) = 0$ , and  $x = 10, x = 2$ . The corresponding values of  $y$  found from the linear equation are 25 and 1.

$\therefore$  The points of intersection are  $(10, 25), (2, 1)$ .

## Exercises

1. Determine if the points (2, 1), (-1, 3), (4, 6) lie on the curves whose equations are:

$$4y^2 = x; \quad x^2 + y^2 = 5; \quad x^2 - y^2 + 8 = 0; \quad y^2 = 9x.$$

Find the points of intersection of the following pairs of loci. Plot the curves and verify your solution.

$$\begin{aligned} 2. \quad & 2x - y = 1, \\ & 3x + 2y = 5. \end{aligned}$$

$$\begin{aligned} 3. \quad & y - x = 0, \\ & 3x + 4y - 7 = 0. \end{aligned}$$

$$\begin{aligned} 4. \quad & y^2 = 4x, \\ & y - x - 1 = 0. \end{aligned}$$

$$\begin{aligned} 5. \quad & x^2 + y^2 = 25, \\ & 4x^2 + 36y^2 = 144. \end{aligned}$$

$$\begin{aligned} 6. \quad & y^2 = 8x, \\ & 2x + y = 8. \end{aligned}$$

$$\begin{aligned} 7. \quad & x^2 + y^2 = 25, \\ & x^2 + 4y^2 = 73. \end{aligned}$$

$$\begin{aligned} 8. \quad & x^2 - y^2 = 7, \\ & y - 2x + 5 = 0. \end{aligned}$$

$$\begin{aligned} 9. \quad & x^2 + y^2 = 25, \\ & x^2 - y^2 = 11. \end{aligned}$$

$$\begin{aligned} 10. \quad & x^2 + y^2 = 5, \\ & xy = 2. \end{aligned}$$

$$\begin{aligned} 11. \quad & 2x^2 + y^2 = 5, \\ & x^2 + 2y^2 = 1. \end{aligned}$$

$$\begin{aligned} 12. \quad & x^4 + y^2 = x^2, \\ & x = 2y. \end{aligned}$$

$$\begin{aligned} 13. \quad & 3x^2 + y^2 = 9, \\ & x^2 - y^2 = 3. \end{aligned}$$

**26. Product of two or more equations.** *Given two or more equations with their second members zero. If the product of the first members be equated to zero, the locus of the new equation is the combined loci of the given equations.*

If all the terms of an equation are transposed to the left-hand side, we often use a single letter to represent the expression in  $x$  and  $y$ . If the expression is represented by  $u$ , the equation is  $u = 0$ .

Suppose there are two given equations and let us represent them by  $u = 0$  and  $v = 0$ . The new equation is

then  $uv = 0$ . It is evident that the coordinates of every point which causes either  $u$  or  $v$  to vanish will satisfy the equation  $uv = 0$ . Moreover, the coordinates of no other point can satisfy  $uv = 0$ . Therefore, the locus of the equation  $uv = 0$  is the loci of the two equations  $u = 0$  and  $v = 0$ .

*Example 1.* Sketch the locus of

$$x^2 - y^2 = 0.$$

Solution:

$$x^2 - y^2 = (x + y)(x - y) = 0.$$

$$\therefore x - y = 0, \quad x + y = 0,$$

whose graphs are given in Fig. 38.

Hence the required locus consists of two lines through the origin and bisecting the angles formed by the axes.

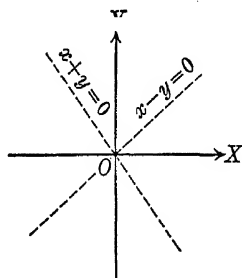


FIG. 38

*Example 2.* Sketch the locus of

$$x^3 + xy^2 - 4x = 0.$$

Solution:

$$x^3 + xy^2 - 4x =$$

$$x(x^2 + y^2 - 4) = 0.$$

$$\therefore x = 0, \quad x^2 + y^2 - 4 = 0,$$

whose graphs are given in Fig 39.

Hence the required locus consists of the  $y$ -axis and a circle center at  $O$  and radius of 2.

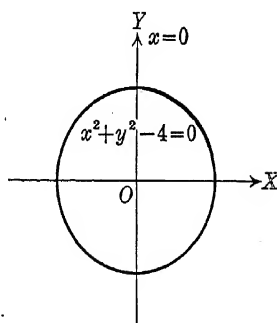


FIG. 39

**Exercises**

Determine the loci of the following equations:

1.  $xy = 0$ .

2.  $y^2 - x^2 = 0$ .

3.  $2x + 3x^2 = 0$ .

4.  $2y^2 + 5y = 0$ .

5.  $x^3 + xy^2 - 9x = 0$ .

6.  $3x^2 - xy - 5x = 0$ .

7.  $x^2y = xy^2$ .

8.  $(x^2 + y^2) - 13(x^2 + y^2) + 36 = 0$ .

9.  $(x^2 + y^2 - 9)(y^2 - 4x) = 0$ .

10.  $x^2 - 3x + 2 = 0$ .

11.  $(y^2 - 2x)(x + y - 2) = 0$ .

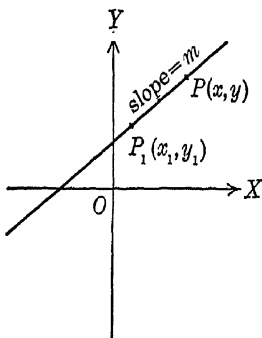
12.  $(x - 3y + 1)(x + 2y) = 0$ .

## CHAPTER IV

### THE STRAIGHT LINE

**27. The point-slope form.** In the last chapter we stated that a very important problem of analytic geometry is to learn how to plot a curve when its equation is given. A second important problem is, given a curve, to find its equation with respect to a given set of coordinate axes. We shall now consider the simplest case of this problem, namely, where the given curve is a straight line.

Let the given line pass through a fixed point  $P_1(x_1, y_1)$  and have a given slope  $m$ . Since the line has a slope we know it is not parallel to the  $y$ -axis. Let  $P(x, y)$  be a variable point on the given line. Then, equating slopes, we have



$$\frac{y - y_1}{x - x_1} = m \quad \text{or}$$

$$(1) \quad y - y_1 = m(x - x_1).$$

FIG. 40

This equation is satisfied by the coordinates of every point on the line and by no other points, since the ratio  $\frac{y - y_1}{x - x_1}$  formed for a point  $(x, y)$  not on the line, does not equal  $m$ .

Equation (1) is called the *point-slope equation* of a straight line. It enables us to write the equation of the

line passing through any given point and having any given slope.

*Example.* Find the equation of the straight line passing through  $(1, -2)$  and having a slope of  $-3$ .

Solution:  $x_1 = 1$ ,  $y_1 = -2$ ,  $m = -3$ .

Substituting in  $y - y_1 = m(x - x_1)$ , we have

$$y + 2 = -3(x - 1),$$

or  $3x + y - 1 = 0.$

**28. Slope-intercept form.** As a special case of the last article, suppose the given point is on the  $y$ -axis and has the coordinates  $(0, b)$ . The equation of the line is then

$$y - b = m(x - 0) \text{ or}$$

$$(2) \quad y = mx + b.$$

The equation of every straight line, except those parallel to the  $y$ -axis, can be written in this form.

From this form we note that if the equation of a line is solved for  $y$ , the coefficient of  $x$  is the slope.

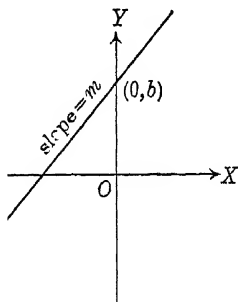


FIG. 41

*Example 1.* Find the slope of the line whose equation is  $3x + 2y - 7 = 0$ .

Solution: Solving for  $y$  we have  $y = -\frac{3}{2}x + \frac{7}{2}$ .

$\therefore$  The slope is  $-\frac{3}{2}$ .

*Example 2.* Find the equation of the straight line through  $(2, 3)$  and perpendicular to  $2x - 3y + 6 = 0$ .

Solution: The slope of the given line is  $\frac{2}{3}$ .

$\therefore$  By § 14 the slope of the required line is  $-\frac{3}{2}$ .

Since the line passes through  $(2, 3)$  its equation is  $y - 3 = -\frac{3}{2}(x - 2)$  or  $3x + 2y - 12 = 0$ .

### Exercises

Find the equations of the following straight lines:

1. Through  $(1, 4)$  with slope 2.
2. Through  $(-1, -3)$  with slope  $-2$ .
3. Through  $(0, -4)$  with slope  $\frac{1}{2}$ .
4. Through  $(1, 5)$  and making an angle of  $45^\circ$  with the  $x$ -axis.
5. Through the origin and with an inclination of  $60^\circ$ .
6. With  $y$ -intercept 4 and slope  $-1$ .
7. With  $y$ -intercept  $-1$  and slope  $\frac{3}{4}$ .
8. With  $y$ -intercept  $b$  and slope  $\frac{a}{b}$ .
9. Through  $(2, -3)$  and having an angle of inclination of  $135^\circ$ .
10. Through points  $(3, 4)$ ,  $(-2, 5)$ . [*Hint:* First find the slope and then apply formula 1.]
11. Through points  $(2, -4)$ ,  $(6, 7)$ .
12. Through points  $(-2, -3)$ ,  $(-4, -5)$ .

Find the slopes of the following lines:

13.  $4x + y = 7$ .
14.  $-3x + 2y = 7$ .

15.  $2x - 3y = 8$ .                      16.  $ax + by = c$ .
17.  $2px - 3qy - r = 0$ .
18.  $\frac{x}{a} + \frac{y}{b} = 1$ .
19.  $x \cos \alpha + y \sin \alpha = p$ .
20. Prove that the equation of a line passing through the origin with slope  $m$  is  $y = mx$ .
21. Which of the following straight lines are parallel and which are perpendicular?
- a)  $2x - y + 3 = 0$ .                      b)  $2x + y + 4 = 0$ .  
c)  $x + 2y - 7 = 0$ .                      d)  $x - 2y + 8 = 0$ .  
e)  $4x - 2y + 5 = 0$ .                      f)  $x - 2y + 4 = 0$ .  
g)  $2x + 4y - 9 = 0$ .                      h)  $2x - y + 8 = 0$ .
22. Find the equation of the straight line through  $(6, -2)$ :  
a) parallel to  $4x - 3y = 7$ .  
b) perpendicular to  $4x - 3y = 7$ .
23. Find the equation of the straight line through  $(3, 6)$ :  
a) parallel to  $2x + 5y = 3$ .  
b) perpendicular to  $-2x + 5y = 7$ .
24. Find the equation of the straight line through  $(a, b)$ :  
a) parallel to  $Ax + By + C = 0$ .  
b) perpendicular to  $Ax + By + C = 0$ .
25. Find the equation of the straight line through  $(6, 5)$ :  
a) parallel to the  $x$ -axis.  
b) perpendicular to the line through  $(3, 4)$ ,  $(-4, 5)$ .
26. Prove that the points  $(1, 2)$ ,  $(3, 8)$ ,  $(-1, -4)$  are collinear.
27. Prove that the points  $(1, -3)$ ,  $(2, -1)$ ,  $(-2, -9)$  are collinear.

28. Find the fourth vertex of a parallelogram three of whose vertices are  $(2, 1)$ ,  $(4, 3)$ ,  $(3, 7)$ . (Three solutions)
29. Three vertices of a parallelogram are  $(-3, 1)$ ,  $(6, -3)$ ,  $(2, 3)$ . Find the fourth vertex if it lies in the second quadrant.
30. The vertices of a triangle are  $(6, 2)$ ,  $(8, -3)$ ,  $(-3, 5)$ . Find the equations of the lines through the vertices parallel to the opposite sides.
31. The vertices of a triangle are  $(6, 2)$ ,  $(-4, 3)$ ,  $(2, 4)$ . Find the equations of the lines through the vertices perpendicular to the opposite sides. Prove these lines are concurrent.

**29. Two-point form.** If two points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  are given on a line not parallel to the  $y$ -axis, the slope of the line is  $m = \frac{y_2 - y_1}{x_2 - x_1}$ . Therefore, the equation of the line from (1) is

$$(3) \quad y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1).$$

If (3) is cleared of fractions we obtain  $xy_1 - xy_2 - yx_1 + yx_2 + x_1y_2 - x_2y_1$  which may be written in the determinant form

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

*Example.* Find the equation of the line through  $(1, -2)$ ,  $(-2, 3)$ .

Solution: The slope  $m = \frac{3 + 2}{-2 - 1} = -\frac{5}{3}$ .

The equation is  $y + 2 = -\frac{2}{3}(x - 1)$ ,

or  $5x + 3y + 1 = 0$ .

(If we had written the equation of the line through  $(-2, 3)$  with slope  $-\frac{5}{3}$ , the same equation would have been obtained. Let the student verify this statement.)

### Exercises

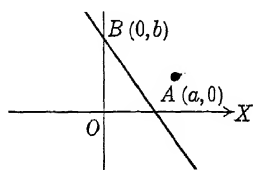
Write the equations of the lines through the following pairs of points:

1.  $(7, 2)$  and  $(3, -4)$ .
2.  $(1, 3)$  and  $(2, 4)$ .
3.  $(-2, 3)$  and  $(4, -1)$ .
4.  $(-1, -1)$  and  $(2, 6)$ .
5. The origin and  $(4, 5)$ .
6. The origin and  $(-1, 4)$ .
7. The origin and  $(-2, -1)$ .
8.  $(a, b)$  and  $(b, a)$ .
9.  $(a, 0)$  and  $(0, b)$ .
10.  $(p, q)$  and  $(m, n)$ .

**30. Intercept form.** Suppose a line cuts the axes in the points  $A(a, 0)$ , and  $B(0, b)$ , where neither  $a$  nor  $b$  is zero, then  $OA = a$  is the  $x$ -intercept and  $OB = b$  is the  $y$ -intercept. The slope of this

line is  $m = -\frac{b}{a}$  and its equation is

$$y - b = -\frac{b}{a}x$$



If we divide both members of this equation by  $b$ , the equation can be written in the form

FIG. 42

$$(4) \quad \frac{x}{a} + \frac{y}{b} = 1, \quad (ab \neq 0)$$

which is called the intercept form of the equation of a straight line.

*Example 1.* Find the equation of the straight line whose  $x$ -intercept is 4 and whose  $y$ -intercept is  $-2$ .

Solution:  $a = 4$ ,  $b = -2$ .  $\therefore$  The required equation is

$$\frac{x}{4} + \frac{y}{-2} = 1.$$

*Example 2.* Write in intercept form the equation  $2x - 3y - 5 = 0$ .

Solution: Transposing the constant term, we have  $2x - 3y = 5$ . Dividing by 5

$$\frac{2x}{5} - \frac{3y}{5} = 1,$$

or 
$$\frac{x}{\frac{5}{2}} + \frac{y}{-\frac{5}{3}} = 1.$$

This equation is in form (4) and  $a = \frac{5}{2}$ ,  $b = -\frac{5}{3}$ .

### Exercises

Write the equation of each of the lines whose intercepts are given below. The  $x$ -intercept in each case is given first.

1. 2, 3.                      2.  $-1, -2$ .                      3.  $\frac{1}{2}, \frac{1}{3}$ .

4.  $3a, 2b$ .                      5.  $2a + b, b - 2a$ .

Write in intercept form the following equations:

6.  $2x - 3y = 16$ .                      7.  $3x + 2y = 12$ .

8.  $4x - 8y = 25$ .                      9.  $-3x + 2y = 17$ .

10. If a line has intercepts  $a$  and  $b$ , and the perpendicular from

the origin to the line is of length  $p$ , and the angle of inclination of  $p$  is  $\alpha$ , prove that the equation of the line is

$$x \cos \alpha + y \sin \alpha = p.$$

**31. Lines parallel to the axes.** Since a line parallel to the  $y$ -axis does not have a slope, the point-slope form and slope-intercept form do not apply. Moreover, since the line has no  $y$ -intercept, the intercept form does not apply. However, the equation of any line parallel to the  $y$ -axis is of the form

$$(5) \quad x = a,$$

where  $a$  is a constant and equal to the abscissa of any point of the line. Thus the equation of the line through the point  $(-2, 7)$  and parallel to the  $y$ -axis is  $x = -2$ .

Similarly, the equation of any line parallel to the  $x$ -axis is of the form

$$(6) \quad y = b,$$

where  $b$  is a constant and equal to the ordinate of any point on the line. In this case the slope is zero so that the point-slope and slope-intercept form can be used. Since the line has no  $x$ -intercept, the intercept form cannot be used.

### Exercises

1. Find the equation of the  $y$ -axis.
2. Find the equation of the  $x$ -axis.
3. Find the equation of the line through  $(2, -1)$  parallel to the  $y$ -axis.
4. Find the equation of the line through  $(2, -1)$  parallel to the  $x$ -axis.

5. Find the equations of the two lines passing through  $(a, a)$  forming with the coordinate axes a square.
6. Find the equation of the line through  $(6, 7)$  perpendicular to the  $x$ -axis;  $y$ -axis.

**32. Normal form.** Let  $AB$  be any straight line with  $OQ$  the perpendicular from the origin upon it. The direction  $OQ$  will be taken as the positive direction of the perpendicular. If the length of  $OQ$  is  $p$ , and its angle of inclination is  $\alpha$ , the equation of  $AB$  in terms of  $p$  and  $\alpha$  is called the **normal form** of the straight line. Let  $P(x, y)$  be any point on  $AB$ . From Theorem 1, § 6, the projection of the broken line  $OMP$  on  $OQ$  is equal to the projection of  $OP$  on  $OQ$ , that is

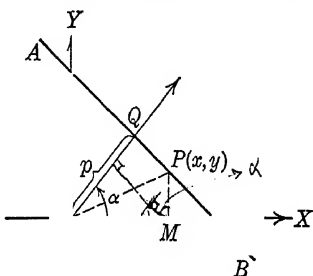


FIG. 43

$$\text{Proj}_{OQ} OM + \text{Proj}_{OQ} MP = \text{Proj}_{OQ} OP.$$

From Theorem 2, § 6, the projection of  $OM$  on  $OQ$ , i.e.,  $\text{Proj}_{OQ} OM = OM \cos \alpha = x \cos \alpha$ .

The projection of  $MP$  on  $OQ$ , i.e.,  $\text{Proj}_{OQ} MP = MP \cos \left( \frac{\pi}{2} - \alpha \right) = MP \sin \alpha = y \sin \alpha$ .

The projection of  $OP$  on  $OQ$ , i.e.,  $\text{Proj}_{OQ} OP = p$ .

$$(7) \quad \therefore x \cos \alpha + y \sin \alpha = p$$

is the desired equation.

Hence the coordinates of any point  $P$  on the line  $AB$  must satisfy equation (7). It is left as an exercise for the student to prove conversely, that if the coordinates of a point satisfy equation (7), the point is on the line  $AB$ .

As previously stated we shall assume the positive direction of  $p$  to be from the origin towards the line. If the line passes through the origin,  $p = 0$  and  $\alpha$  is taken to be less than  $180^\circ$ .

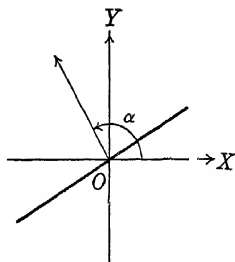


FIG. 44

### Exercises

1. The perpendicular from the origin to a line is 4 and the perpendicular makes an angle of  $\frac{\pi}{4}$  with the  $x$ -axis; find the equation of the line.
2. A line is 10 units from the origin and makes an angle of  $120^\circ$  with the  $x$ -axis; find its equation.
3. If  $\alpha = 45^\circ$ , find  $p$  in order for the line  $x \cos \alpha + y \sin \alpha = p$  to pass through the point  $(3, 4)$ .
4. A line passes through  $(-8, -6)$  and the perpendicular from the origin to the line makes an angle of  $225^\circ$  with the  $x$ -axis; find the equation of the line.
5. Two parallel lines are on the same side of the origin. Do they have the same angle  $\alpha$ ?
6. Two parallel lines are on opposite sides of the origin. Do they have the same angle  $\alpha$ ?
7. A line of slope 1 passes through the origin. Find  $\alpha$  and  $p$ .

8. In the equation  $x \cos \alpha + y \sin \alpha = p$ ,  $p$  increases while  $\alpha$  remains constant. What is the effect upon the line? If  $\alpha$  varies while  $p$  remains constant, what is the effect?

**33. The general equation of the first degree.** Any line drawn in the plane will either cut the  $y$ -axis or be parallel to it. If it cuts the  $y$ -axis its equation is of the form  $y = mx + b$ , where  $m$  is its slope and  $b$  its  $y$ -intercept. If the line is parallel to the  $y$ -axis its equation is of the form  $x = a$ , where  $a$  is its  $x$ -intercept. Hence we have the

*Theorem: The equation of every straight line is an equation of the first degree in  $x$  and  $y$ .*

The most general equation of the first degree in  $x$  and  $y$  is of the form

$$(8) \quad Ax + By + C = 0,$$

where  $A$ ,  $B$ , and  $C$  are constants.

The question now arises, is this always the equation of a straight line, no matter what the numbers  $A$ ,  $B$ , or  $C$  may be?

We can assume that  $A$  and  $B$  are not both zero; for if they were, the equation would no longer contain either  $x$  or  $y$  and so would not be of the first degree. We shall consider the two cases: a)  $B \neq 0$ ; b)  $B = 0$ .

a) If  $B \neq 0$ , we can solve the equation for  $y$ :

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

Comparison with (2) shows that this is the equation of a line whose slope is  $-\frac{A}{B}$  and whose  $y$ -intercept is  $-\frac{C}{B}$ .

Such a line exists.

b) If  $B = 0$ , the equation is of the form  $Ax + C = 0$ . Since we know that in this case  $A \neq 0$ , we can write this equation in the form

$$x = -\frac{C}{A}.$$

This is the equation of a straight line parallel to the  $y$ -axis.

**Theorem:** Every equation of the form  $Ax + By + C = 0$  (in which  $A$  and  $B$  are not both zero) is the equation of a straight line. If  $B \neq 0$ , its slope is  $-\frac{A}{B}$ , and its  $y$ -intercept is  $-\frac{C}{B}$ ; if  $B = 0$ , it is parallel to the  $y$ -axis; if  $A = 0$ , it is parallel to the  $x$ -axis; if  $C = 0$ , it passes through the origin.

The student should prove the last two statements of the theorem.

### 34. Reduction of $Ax + By + C = 0$ to special forms.

#### 1. Slope-intercept form.

Suppose  $B \neq 0$ . Then, solving for  $y$ , we have

$$y = -\frac{A}{B}x + \frac{-C}{B}.$$

This is in the form  $y = mx + b$ , where

$$m = -\frac{A}{B}, \quad b = -\frac{C}{B}.$$

If  $B = 0$ , the equation cannot be reduced to the slope-intercept form. Why? The line in this case is parallel to the  $y$ -axis.

2. *Intercept form.*

Suppose  $A$ ,  $B$ , and  $C$  are not 0. It is left as an exercise to show the equation can be written

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1.$$

This equation is in the form  $\frac{x}{a} + \frac{y}{b} = 1$  where

$$a = -\frac{C}{A}, \quad b = -\frac{C}{B}.$$

If either  $A$ ,  $B$ , or  $C$  is 0, the equation cannot be written in intercept form. Why? The line in this case is either parallel to an axis or passes through the origin.

3. *Normal form.*

Since  $\sin^2 \alpha + \cos^2 \alpha = 1$ , it is evident that we must multiply all the coefficients of

$$Ax + By + C = 0$$

by a number  $k$ , so chosen that  $(kA)^2 + (kB)^2 = 1$ . This condition will be satisfied if

$$k = \pm \frac{1}{\sqrt{A^2 + B^2}}.$$

Hence to reduce  $Ax + By + C = 0$  to the normal form, divide every term by

$$\pm \sqrt{A^2 + B^2}, \text{ giving}$$

$$\frac{A}{\pm \sqrt{A^2 + B^2}} x + \frac{B}{\pm \sqrt{A^2 + B^2}} y + \frac{C}{\pm \sqrt{A^2 + B^2}} = 0.$$

The sign of the radical is chosen as follows:

1. If  $C \neq 0$ , then since  $p > 0$ , the sign of the radical must be chosen opposite to that of  $C$ .
2. If  $C = 0$ , then since  $\alpha < 180^\circ$ ,  $\sin \alpha > 0$ , the sign of the radical must be chosen the same as  $B$ . Since  $A$  and  $B$  cannot both be zero,  $A^2 + B^2$  cannot be zero and hence every equation of a line can be reduced to the normal form.

*Example.* Reduce the equation  $5x - 12y + 39 = 0$ ,  
a) to slope form, b) intercept form, c) normal form.

Solution: a) Solving for  $y$ , we have

$$y = \frac{5}{12}x + \frac{39}{12}.$$

b) The intercepts are  $a = -\frac{39}{5}$ ,  $b = \frac{39}{12}$ .

$\therefore$  The desired equation is

$$\frac{x}{-\frac{39}{5}} + \frac{y}{\frac{39}{12}} = 1.$$

[See Ex. 2, § 30.]

c) Dividing each term by  $-\sqrt{5^2 + 12^2} = -13$   
gives

$$-\frac{5}{13}x + \frac{12}{13}y - 3 = 0.$$

### Exercises

Reduce the following equations to a) the intercept form,  
b) the slope-intercept form, c) the normal form.

$$1. 4x - 3y + 7 = 0. \quad 2. 3x + 4y - 5 = 0.$$

$$3. 5x - 4y + 1 = 0. \quad 4. x = 3y - 5.$$

$$5. 2x - y = 0. \quad 6. px + qy = 0 \quad p > 0, \quad q < 0.$$

7. Find the equations of the lines through  $(8, 2)$  and at a distance of  $\sqrt{34}$  units from the origin.
8. Find the equations of the lines parallel to  $4x - 3y = 7$  and at a distance of 2 units from it.
9. Find the equations of the lines parallel to  $5x + 12y - 7 = 0$  and at a distance of 4 units from it.
10. Find in normal form the equation of the line passing through  $(3, 4)$  and perpendicular to the line connecting the point to the origin.

### 35. General theory of parallels and perpendiculars.

*Theorem 1.* Two lines are parallel if their equations differ or may be made to differ only in their constant terms.

Consider the two lines whose equations are

$$(9) \quad Ax + By + C = 0,$$

$$(10) \quad kAx + kBy + C' = 0. \quad (k \neq 0.)$$

Since  $k \neq 0$ , the last equation can be written in the form

$$(11) \quad Ax + By + K = 0, \quad \text{where } K = \frac{C'}{k}.$$

If  $B \neq 0$ , the slope of each of the lines (9) and (11) is  $-\frac{A}{B}$ , that is to say, the lines are parallel.

If  $B = 0$ , then  $A$  cannot be zero (Why?) and the lines are parallel to the  $y$ -axis and hence parallel to each other.

**Theorem 2.** *Two lines are perpendicular if the coefficients of  $x$  and  $y$  in the equation of the first line are equal to or can be made equal to the coefficients of  $-y$  and  $x$ , respectively, in the equation of the second line.*

Consider the two lines whose equations are

$$(12) \quad Ax + By + C = 0,$$

$$(13) \quad kBx - kAy + C' = 0, \quad k \neq 0.$$

Since  $k \neq 0$ , the last equation can be written in the form

$$(14) \quad Bx - Ay + K = 0, \quad \text{where } K = \frac{C'}{k}.$$

If  $B \neq 0$  and  $A \neq 0$ , the slopes of lines (12) and (14) are  $-\frac{A}{B}$  and  $\frac{B}{A}$  respectively. Hence the lines are perpendicular. If  $B = 0$ ,  $A$  cannot be zero. Why? The equation of the first line is  $Ax + C = 0$ , which is the equation of a line perpendicular to the  $x$ -axis. The equation of the second line is  $-Ay + K = 0$ , which is the equation of a line perpendicular to the  $y$ -axis. Hence the two lines are perpendicular to each other. The case  $A = 0$ ,  $B \neq 0$  is left as an exercise.

*Example 1.* Find the equation of the straight line through  $(3, 1)$  parallel to  $2x - 3y = 7$ .

Solution: Any line parallel to the given line is of the form

$$2x - 3y = k.$$

We wish the line that passes through  $(3, 1)$ . There-

fore the coordinates of the point must satisfy the equation. Hence

$$6 - 3 = k \text{ or } k = 3.$$

Therefore the required equation is  $2x - 3y = 3$ .

*Example 2.* Find the equation of the straight line through  $(3, 1)$  perpendicular to  $5x - 2y = 7$ .

Solution: Any line perpendicular to the given line is of the form

$$2x + 5y = k.$$

We wish the line that passes through  $(3, 1)$ . Therefore the coordinates of the point must satisfy the equation. Hence

$$6 + 5 = k \text{ or } k = 11.$$

Hence the required equation is

$$2x + 5y = 11.$$

### Exercises

Using the method of § 35 find the equation of the line through:

1.  $(1, 4)$  parallel to  $2x + 3y = 7$ .
2.  $(2, 7)$  parallel to  $3x - 5y = 6$ .
3.  $(-2, 8)$  parallel to  $6x - 2y = 1$ .
4.  $(4, 7)$  perpendicular to  $2x + 3y = 7$ .
5.  $(6, 4)$  perpendicular to  $8x - 9y = 4$ .
6.  $(-2, -3)$  perpendicular to  $2y - 3x = 6$ .
7. Find the equation of the line parallel to  $3x + 4y = 7$  with  $x$ -intercept 7.

8. Find the equation of the line perpendicular to  $3x + 4y = 7$  with  $y$ -intercept 8.
9. Find the equation of the line perpendicular to  $5x - 6y = 3$  with  $x$ -intercept 3.
10. Find the equation of the line perpendicular to  $5x - 6y = 3$  with  $y$ -intercept  $5/2$ .
11. Find the equation of the line perpendicular to  $2x - 3y = 12$  and passing through the point midway between the two points in which the given line meets the coordinate axes.

**36. Concurrent lines.** If two or more lines meet in a point they are said to be concurrent. Hence, the coordinates of the point of intersection of any two of the lines must satisfy the equations of the other lines.

Let the equations of three concurrent lines be

$$\begin{aligned}
 &A_1x + B_1y + C_1 = 0, \\
 (15) \quad &A_2x + B_2y + C_2 = 0, \\
 &A_3x + B_3y + C_3 = 0.
 \end{aligned}$$

Since the lines are concurrent, the three equations have a single solution. Hence, from § 3

$$(16) \quad \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0.$$

One should note that, conversely, if the determinant (16) is zero, the three lines (15) are not necessarily concurrent. For, if the determinant is zero and

$$A_1 = K_1A_2 = K_2A_3, \quad B_1 = K_1B_2 = K_2B_3,$$

the lines are parallel.

*Example.* Determine  $k$  so that  $x + 2y - 3 = 0$ ,  $4x - y - 3 = 0$ ,  $kx + y + 7 = 0$  are concurrent.

*Solution:*

*Method I.* Solving  $x + 2y - 3 = 0$ ,  $4x - y - 3 = 0$  simultaneously, we find  $x = 1$ ,  $y = 1$ . Substituting these values in  $kx + y + 7 = 0$ , we have  $k + 1 + 7 = 0$  or  $k = -8$ .

*Method II.* By determinants, we have

$$\begin{vmatrix} 1 & 2 & -3 \\ 4 & -1 & -3 \\ k & 1 & 7 \end{vmatrix} = 0,$$

which expanded gives  $k = -8$ .

### Exercises

1. Are the following lines concurrent,  $2x + y - 7 = 0$ ,  $3x + 4y + 6 = 0$ ,  $2x - y - 7 = 0$ ?
2. Find  $k$  so that  $3x - y - 2 = 0$ ,  $2x + y + 1 = 0$ ,  $kx + y + 3 = 0$  are concurrent.
3. Do the following lines form a triangle,  $3x + 4y = -2$ ,  $x + 3y - 2 = 0$ ,  $2x + 5y - 11 = 0$ ?
4. Find  $k$  so that the following lines are concurrent  
 $kx - y = -4$ ,  $2x + 3y = -k$ ,  $x + 2y = -3$ .
5. Prove that the following lines are concurrent

$$x - y - 2 = 0,$$

$$3x + y - 2 = 0,$$

$$4x + 7y + 3 = 0,$$

$$6x - 2y - 8 = 0.$$

**37. The angle between two lines.** Let  $l_1$  and  $l_2$  be two lines intersecting at  $P$  with slopes  $m_1 = \tan \alpha$ ,  $m_2 = \tan \beta$  respectively. The smallest positive angle  $\theta$  through which  $l_1$  must be revolved about  $P$  so as to coincide with  $l_2$ , is called the angle from  $l_1$  to  $l_2$ . There are two cases which must be considered, namely  $\alpha < \beta$  and  $\alpha > \beta$ .

Case 1.  $\alpha < \beta$

$$\theta = \beta - \alpha.$$

$$\therefore \tan \theta = \tan (\beta - \alpha),$$

$$\frac{\tan \beta - \tan \alpha}{1 + \tan \beta \tan \alpha}$$

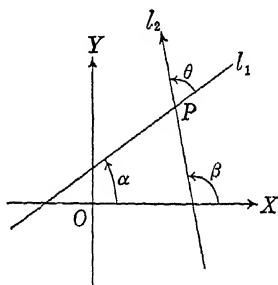


FIG. 45

Case 2.  $\alpha > \beta$

$$180^\circ - \theta = \alpha - \beta.$$

$$\therefore \tan (180^\circ - \theta) = \tan (\alpha - \beta).$$

$$-\tan \theta = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

$$\text{or } \tan \theta = \frac{\tan \beta - \tan \alpha}{1 + \tan \alpha \tan \beta}$$

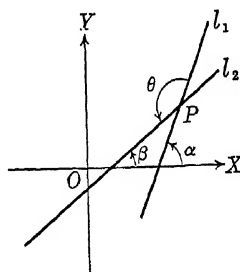


FIG. 45 a

Hence in either case

$$(17) \quad \tan \theta = \frac{m_2 - m_1}{1 + m_2 m_1}.$$

As special cases of this formula we obtain the familiar conditions for parallelism and perpendicularity. If the

lines are parallel,  $\theta = 0^\circ$  or  $180^\circ$ ; since  $\tan \theta$  is zero in either case,  $m_2 - m_1 = 0$  or  $m_1 = m_2$ . If the lines are perpendicular,  $\theta = 90^\circ$  or  $270^\circ$ . But in either case  $\cot \theta$  is zero; therefore  $1 + m_1 m_2 = 0$  or  $m_1 = -\frac{1}{m_2}$ .

In deriving formula (17) we assumed each line had a slope, that is to say, neither line is perpendicular to the  $x$ -axis. If such is not the case, the angle can be readily found without the use of a formula.

*Example 1.* Find the tangent of the angle measured from  $2x + 3y - 6 = 0$  to  $3x - 4y + 12 = 0$ .

Solution:  $m_1 = -\frac{2}{3}$ ,  $m_2 = \frac{3}{4}$ .

$$\therefore \tan \theta = \frac{\frac{3}{4} - (-\frac{2}{3})}{1 + (\frac{3}{4})(-\frac{2}{3})} = \frac{17}{6}.$$

If we wish to find the angle  $\theta$ , we can obtain it from a table of Trigonometric Functions.

*Example 2.* Find the angle from  $x = 3$  to  $x + y = 5$ .

Solution:  $\alpha = 90^\circ + \theta$ .

$$\begin{aligned}\therefore \tan \alpha &= \tan (90^\circ + \theta) \\ &= -\cot \theta.\end{aligned}$$

$\tan \alpha$  is the slope of  $x + y = 5$ , i.e.,  $\tan \alpha = -1$ .

$$\begin{aligned}\therefore -1 &= -\cot \theta, \\ \text{or } \cot \theta &= 1, \text{ and } \theta = 45^\circ.\end{aligned}$$

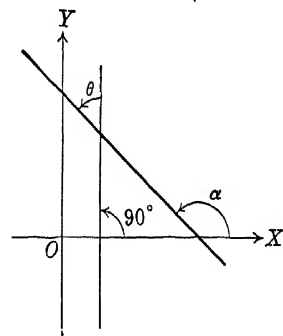


FIG. 46

*Example 3.* Find the equation of the line through  $(2, 3)$  and making an angle of  $60^\circ$  measured from the line whose equation is  $x - 2y = 4$ .

Solution: We have  $\tan \theta = \tan 60^\circ = \sqrt{3}$  and  $m_1 = \frac{1}{2}$ . The slope  $m_2$  of the desired line is obtained from the relation,

$$\tan \theta = \sqrt{3} = \frac{m_2 - \frac{1}{2}}{1 + \frac{1}{2}m_2}$$

$$\frac{2m_2}{2 + m_2}$$

This gives

$$2\sqrt{3} + m_2\sqrt{3} = 2m_2 - 1,$$

and

$$m_2 = \frac{2\sqrt{3} + 1}{2 - \sqrt{3}} = \frac{2\sqrt{3} + 1}{2 - \sqrt{3}} \cdot \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = 5\sqrt{3} + 8.$$

Hence the desired equation is

$$y - 3 = (5\sqrt{3} + 8)(x - 2).$$

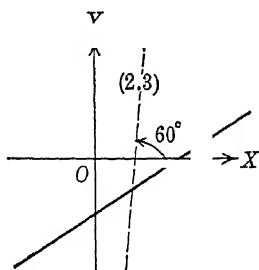


FIG. 47

### Exercises

Find the tangent of the angle from the first line to the second.

- $3x + 4y = 6, 6x - 7y = 1.$
- $x + y = 1, x + 2y = 1.$
- $x - y = 1, 2x - 3y = 5.$
- $ax - by = c, 2ax - by = d.$
- $2x - 3y = 0, x = 4.$
- $3x + y = 7, x = -2.$
- $x = 3, 2x - y = 4.$
- $y = 4, x + y = 8.$
- $y = 5, 2x - y = 4.$
- $x = 7, y = 5.$

Find the equation of the line through  $P$ , making an angle  $\theta$  measured from line  $l$ , when:

11.  $P = (6, -5)$ ,  $\theta = 45^\circ$ ,  $l = 2x + y = 1$ .

12.  $P = (-3, 2)$ ,  $\theta = 135^\circ$ ,  $l = 3x - 2y = 4$ .

13.  $P = (6, 2)$ ,  $\theta = 60^\circ$ ,  $l = 2x - y = 7$ .

14.  $P = \left(\frac{a}{2}, -\frac{b}{2}\right)$ ,  $\theta = \tan^{-1} \frac{b}{a}$ ,  $l = \frac{x}{a} - \frac{y}{b} = 1$ .

15. Find the equations of the lines through  $(2, 3)$  forming with  $2x - y = 4$  an isosceles triangle each of whose base angles contains  $45^\circ$ .

16. Find the equations of the lines through  $(8, -5)$  forming with  $2x - y = 4$  an equilateral triangle.

Find the slopes of the lines bisecting the angles formed by the following pairs of lines:

17.  $3x - 4y = 4$ ,  $4x + 3y - 1 = 0$ .

18.  $3x + 2y - 6 = 0$ ,  $3x - 2y + 5 = 0$ .

19.  $4x - y + 1 = 0$ ,  $x + 4y - 3 = 0$ .

20. Find the tangents of the exterior angles of the triangle formed by the lines whose equations are  $2x - y - 1 = 0$ ,  $3x + y + 10 = 0$ ,  $y = 2$ .

### MISCELLANEOUS EXERCISES

1. Find the intercepts which the line through  $(2, -4)$  and  $(3, 2)$  makes on the axes.

2. What relation exists, if any, between the following pairs of lines?

a)  $3x - y = 4$ ,  $6x - 2y = 5$ ;

b)  $2x + 3y = -7$ ,  $-3x + 2y = 11$ ;

c)  $y = 4x + 8$ ,  $y = -4x + 8$ ;

d)  $y = 3x - 2$ ,  $y = -3x - 2$ .

3. Show that the three points  $(0, 7)$ ,  $(2, -1)$ , and  $(3, -5)$  are collinear, *i.e.*, lie on the same line.
4. Show that the three points  $(3a, 0)$ ,  $(0, 3b)$ , and  $(a, 2b)$  are collinear.
5. What must be the value of  $a$  in order that  $x - 2y + 1 = 0$ ,  $3x + 2y - 5 = 0$ , and  $ax - 4y + 7 = 0$  be concurrent?
6. In what ratio does the line joining the points  $(2, 3)$  and  $(5, 4)$  divide the segment from  $(3, 4)$  to  $(5, 2)$ ?
7. Prove that the three points  $(m, n)$ ,  $(n, m)$ ,  $(-m, 2m + n)$  are collinear.
8. Find the condition that  $(c, d)$ ,  $(d, c)$ , and  $(2c, -d)$  be collinear, provided  $c \neq d$ .
9. Find by two methods the equation of the line through  $(3, 1)$  parallel to the line whose equation is  $3x - 2y - 5 = 0$ .
10. Find by two methods the equation of the line through  $(1, -3)$  parallel to the line joining  $(-2, 3)$  and  $(3, 4)$ .
11. Find by two methods the equation of the line through  $(-2, 3)$  perpendicular to  $4x - 3y = 7$ .
12. Find by two methods the equation of the line through  $(2, 0)$  perpendicular to  $3x + 2y = 5$ .
13. The intercepts of a straight line through  $(-1, 4)$  are equal. Find its equation. [*Hint*: In the intercept form let  $b = a$  and then determine  $a$  so that the line passes through the given point.]
14. Find the equation of the line through  $(6, -5)$  whose  $x$ -intercept is twice the  $y$ -intercept.

15. One side of an isosceles right triangle lies along the  $x$ -axis and the vertex of the right angle is at  $(-2, 4)$ . Find the equations of the other two sides and the coordinates of the other two vertices.
16. Find the equations of the medians of the triangle whose vertices are  $(6, 8)$ ,  $(4, 10)$ ,  $(-2, 6)$  and show that they are concurrent.
17. Find the equations of the medians of the triangle whose vertices are  $(6, 8)$ ,  $(2, 8)$ ,  $(4, -4)$  and show that they are concurrent.
18. Find the equations of the altitudes of the triangle in Ex. 16 and show that they are concurrent.
19. Find the equations of the altitudes of the triangle in Ex. 17 and show that they are concurrent.
20. Find the equations of the perpendicular bisectors of the sides of the triangle in Ex. 16 and show that they are concurrent.
21. Find the equations of the perpendicular bisectors of the sides of the triangle in Ex. 17 and show that they are concurrent.
22. One side of an equilateral triangle is along the  $x$ -axis and the opposite vertex is at  $(3, 6)$ . Find the equations of the other two sides and the coordinates of the other two vertices. [*Hint*: The inclinations of the other two sides can be chosen  $60^\circ$  and  $120^\circ$ , respectively.]

In each of the following cases find the tangent of the angle from the first line to the second:

23.  $2x - y - 6 = 0$ ,  $2x + 3y - 12 = 0$ .
24.  $4x - 5y + 10 = 0$ ,  $x + y - 4 = 0$ .

25.  $3x - 2y + 8 = 0$ ,  $2x + 3y - 9 = 0$ .
26.  $2x + y + 2 = 0$ ,  $3x - 2y - 7 = 0$ .
27. Find the equation of the line through the point  $(2, -1)$  and making an angle of  $45^\circ$  with the line  $y = 2x$ .

**38. Distance from a line to a point.** Let the given line  $AB$  have the equation  $x \cos \alpha + y \sin \alpha = p$  and the given point  $P_1$  the coordinates  $(x_1, y_1)$ . We wish to

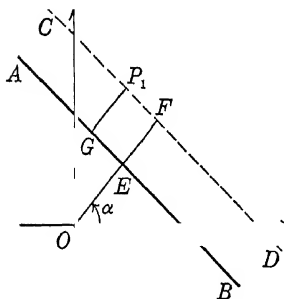


FIG. 48

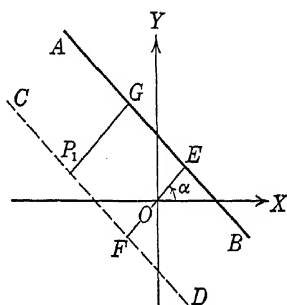


FIG. 49

find the distance  $GP_1$ , measured from  $AB$  to  $P_1$ . Through  $P_1$  draw a line  $CD$  parallel to  $AB$ . Its equation is

$$x \cos \theta + y \sin \theta = p_1,$$

where  $\theta = \alpha$  or  $180^\circ + \alpha$ . Hence, the equation of  $CD$  is  $x \cos \alpha + y \sin \alpha = \pm p_1$ . If  $CD$  is on the same side of the origin as  $AB$ ,  $p_1$  is positive, while if it is on the opposite side of the origin from  $AB$ ,  $p_1$  is negative.

In all cases

$$GP_1 = EF = OF - OE = p_1 - p.$$

Since  $P_1(x_1, y_1)$  is on  $CD$ , its coordinates must satisfy the equation of  $CD$ , i.e.,

$$x_1 \cos \alpha + y_1 \sin \alpha = p_1$$

$$\therefore GP_1 = p_1 - p = x_1 \cos \alpha + y_1 \sin \alpha - p.$$

*Hence to find the distance from a line to a point reduce the equation of the line to the normal form with all terms transposed to the left-hand member; substitute the coordinates of the given point in the left member. The result is the required distance.*

Thus, if the given equation is  $Ax + By + C = 0$  and the given point is  $(x_1, y_1)$ ,

$$d = \frac{Ax_1}{\pm\sqrt{A^2+B^2}} + \frac{By_1}{\pm\sqrt{A^2+B^2}} + \frac{C}{\pm\sqrt{A^2+B^2}}$$

or

$$(18) \quad d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2+B^2}}.$$

The sign of the radical is chosen opposite to the sign of  $C$  if  $C \neq 0$  and the same as the sign of  $B$  if  $C = 0$ .

**39. Sides of a line.** In deriving formula (18) the distance was measured *from* the line *to* the point. Hence if two points  $P_1$  and  $P_2$  are on the same side of a line, the distances  $d_1$  and  $d_2$  have the same sign, while if they are on opposite sides of the line, the distances  $d_1$  and  $d_2$  have opposite signs.

The direction  $OQ$  was chosen as positive (§ 32) and hence if  $C \neq 0$ , formula (18) will be positive if  $P_1$  and the origin are on opposite sides of the line and negative if  $P_1$

and the origin are on the same side of the line. Therefore, the side of the line on which the origin lies is called the *negative side* and the other side, the *positive side*.

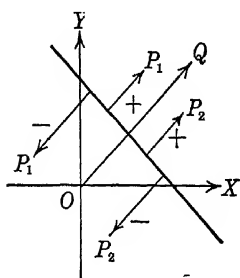


FIG. 50

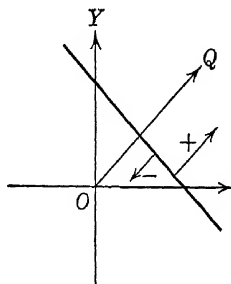


FIG. 51

If the line passes through the origin, *i.e.*,  $C = 0$ , formula (18) will give a positive value for  $d$  if  $P_1$  is *above the line* and a negative value if  $P_1$  is *below the line*.

*Example 1.* Find the distance from the line

$$2x - 3y = 5 \text{ to the point } (-2, 1).$$

*Solution:* Applying formula (18) we have

$$d = \frac{2(-2) - 3(1) - 5}{\sqrt{4 + 9}} = -\frac{12}{\sqrt{13}}.$$

Since  $d$  is negative the given point and the origin are on the same side of the line.

*Example 2.* Find the length of the altitude drawn from the vertex  $A$  in the triangle whose vertices are  $A(3, 4)$ ,  $B(-2, 5)$ ,  $C(-4, -4)$ .

*Solution:* In order to find the distance from the line  $BC$  to the point  $A$ , we must know the equation of  $BC$ . The

slope of  $BC$  is  $\frac{3}{2}$  and its equation is  $y + 4 = \frac{3}{2}(x + 4)$  or  $9x - 2y + 28 = 0$ . Hence

$$d = \frac{9(3) - 2(4) + 28}{-\sqrt{81 + 4}} = -\frac{47}{\sqrt{85}}.$$

An altitude of a triangle is an unsigned number,

$$\therefore \text{altitude} = \frac{47}{85} \sqrt{85}.$$

*Example 3.* Find the equations of the bisectors of the angles between the lines  $3x - 4y + 12 = 0$  and  $5x + 12y - 60 = 0$ .

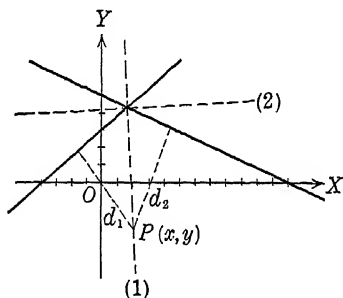


FIG. 52

*Solution:* First draw the lines. From plane geometry we know that the bisector of an angle is the locus of points equidistant from the sides of the angle. Let  $P(x, y)$  be any point on bisector (1). From inspection we see  $P$  is either on the positive side or negative side of both lines. Therefore we have  $d_1 = d_2$ , or

$$\frac{3x - 4y + 12}{-5} = \frac{5x + 12y - 60}{13}$$

or  $8x + y - 18 = 0.$

If  $P$  is on bisector (2) it is on the positive side of one line and on the negative side of the other. Therefore

$$d_1 = -d_2, \text{ or } \frac{3x - 4y + 12}{-5} = -\frac{5x + 12y - 60}{13}$$

$$\text{or } 7x - 56y + 228 = 0.$$

### Exercises

In each of the following examples find the distance of the given point from the given line:

1.  $(2, 4)$ ,  $3x - 4y + 12 = 0$ .

2.  $(-1, -2)$ ,  $4x + 3y - 8 = 0$ .

3.  $(-3, 4)$ ,  $-2x + y - 8 = 0$ .

In each of the following examples find the lengths of the altitudes of the triangle whose vertices are the given points:

4.  $(2, 1)$ ,  $(-1, 3)$ ,  $(-2, -4)$ .

5.  $(0, 4)$ ,  $(4, -1)$ ,  $(1, -3)$ .

6.  $(-1, -2)$ ,  $(-3, 1)$ ,  $(2, -3)$ .

7.  $(0, 3)$ ,  $(5, 0)$ ,  $(3, 6)$ .

In each of the following cases find the distance between the two given parallel lines:

8.  $3x + 4y - 12 = 0$ ,  $6x + 8y - 48 = 0$ .

9.  $5x - 12y + 13 = 0$ ,  $5x - 12y + 26 = 0$ .

10.  $2x - y + 4 = 0$ ,  $4x - 2y + 5 = 0$ .

11.  $x + y - 3 = 0$ ,  $x + y + 7 = 0$ .

12. Prove that the distance between any two parallel lines whose equations are  $ax + by + c = 0$  and  $ax + by + c' = 0$  is in absolute value equal to  $\frac{|c - c'|}{\sqrt{a^2 + b^2}}$ .
13. Find the coordinates of the two points on the  $x$ -axis which are a distance 4 from the line whose equation is  $2x + 3y - 4 = 0$ .
14. Find the coordinates of the two points on the  $y$ -axis which are a distance 2 from the line whose equation is  $3x + 4y - 12 = 0$ .
15. Find the coordinate of the point on the  $x$ -axis which is equidistant from the points (2, 6) and (-3, 4).
16. Find the equation of a line parallel to the line whose equation is  $3x + 4y - 10 = 0$  and at a distance 2 from it. (Two solutions.)
17. Find the equation of a line parallel to the line whose equation is  $x + y - 1 = 0$  and at a distance 3 from it. (Two solutions.)
18. *Derive the formula for the distance of a point  $(x_1, y_1)$  from the line  $Ax + By + C = 0$  by finding the intersection of the perpendicular through the given point and the given line, and then using the formula for the distance between two points.*
19. A straight line moves so that the sum of the reciprocals of its intercepts on the axes is constant. Prove that the line passes through a fixed point.
20. Find the equations of the bisectors of the acute angles formed by the following lines:
- a)  $3x + 4y - 7 = 0$ ,  $4x + 3y - 12 = 0$ .
- b)  $5x + 12y - 7 = 0$ ,  $12x + 5y + 20 = 0$ .

21. Find the equations of the bisectors of the obtuse angles formed by the following lines:

a)  $2x + 3y - 7 = 0$ ,  $3x + 2y + 10 = 0$ .

b)  $3x - 4y - 8 = 0$ ,  $5x + 12y - 20 = 0$ .

22. Find the equations of the bisectors of the angles of the following triangles and prove that they are concurrent:

a)  $3x + y - 2 = 0$ ,  $x - 3y - 6 = 0$ ,  $x + 3y + 12 = 0$ .

b)  $x + 4 = 0$ ,  $y - 6 = 0$ ,  $y + x = 0$ .

23. The equations of the sides of a triangle are  $5x + 12y - 12 = 0$ ,  $5x - 12y = 0$  and  $12x + 5y + 30 = 0$ . Show that the bisector of the interior angle at the vertex formed by the first two lines and the bisectors of the exterior angles at the other vertices are concurrent.

24. Find the locus of a point the ratio of whose distances from the lines  $3x - 4y - 12 = 0$  and  $5x + 12y - 24 = 0$  is 12:7.

**40. Area of a triangle.** Let a triangle be determined by the points  $P_1(x_1, y_1)$ ,  $P_2(x_2, y_2)$ ,  $P_3(x_3, y_3)$ . Draw the perpendicular  $P_3S$  to the side  $P_1P_2$ . Then  
 Area of  $\triangle = \frac{1}{2} P_1P_2 \times P_3S$ .  
 From § 10 it follows that

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

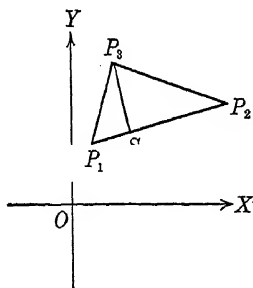


FIG. 53

From § 29 it follows that the equation of  $P_1P_2$  can be written in the form

$$(y_2 - y_1)x - (x_2 - x_1)y - x_1y_2 + x_2y_1 = 0.$$

Therefore, from § 38 it follows that

$$P_3S = \frac{(y_2 - y_1)x_3 - (x_2 - x_1)y_3 - x_1y_2 + x_2y_1}{\pm \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}.$$

Hence the area  $A$  of the triangle is

$$A = \pm \frac{1}{2} [(y_2 - y_1)x_3 - (x_2 - x_1)y_3 - x_1y_2 + x_2y_1]$$

which simplifies to

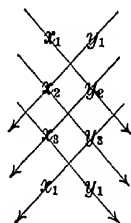
$$(19) A = \pm \frac{1}{2} [(x_1 - x_2)y_3 + (x_2 - x_3)y_1 + (x_3 - x_1)y_2].$$

In determinant notation this formula may be written

$$A = \pm \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Since we are interested only in the numerical value of the area, the sign is chosen to make  $A$  positive.

The following scheme makes it easy to remember formula (19). Write the coordinates of the vertices in two vertical columns, repeating the coordinates of the first vertex. First, multiply each  $x$  by the  $y$  in the next row and add the products which gives  $x_1y_2 + x_2y_3 + x_3y_1$ . Second, multiply each  $y$  by the  $x$  in the next row and add the products which gives  $y_1x_2 + y_2x_3 + y_3x_1$ .



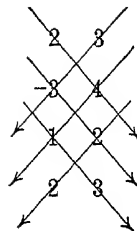
Third, the area is equal to  $\pm \frac{1}{2}$  the difference between the first and second sums, the sign to be chosen to give  $A$  a positive value.

*Example.* Find the area of the triangle whose vertices are

$$(2, 3), (-3, 4), (1, 2).$$

Solution: Write the coordinates of the vertices in two vertical columns, repeating the first vertex. Performing the first step described above, we have  $8 - 6 + 3 = 5$ ; the second step gives  $-9 + 4 + 4 = -1$ . Hence

$$A = \frac{1}{2} (5 + 1) = 3.$$



### Exercises

Find the area of the following triangles:

- $(2, 4), (-1, 0), (3, 2)$ .
- $(1, 0), (4, 0), (5, 2)$ .
- $(a, a), (c, d), (-b, -b)$ .
- $(0, 1), (-1, -1), (2, 5)$ .
- $(0, -p), (1, m-p), (m, m^2-p)$ .
- Find the area of the quadrilateral  $(3, 4), (-2, 1), (3, 0), (-2, -4)$ .
- Find the area of the pentagon  $(0, 0), (1, 0), (4, 2), (3, 7), (-2, 1)$ .
- Prove the following points collinear by proving that the area of the triangle whose vertices are the given point is zero:
  - $(0, -2), (2, 4), (-1, -5)$ .
  - $(p, p+q), (-q, 0), (-p, q-p)$ .
  - $(a, b+c), (b, c+a), (c, a+b)$ .
- Prove that the area of the triangle whose vertices are  $(8, 6), (-2, 4), (4, -4)$  is four times the area of the triangle formed by joining the middle points of the sides.

10. Given the quadrilateral  $A(0, 0)$ ,  $B(2, 0)$ ,  $C(6, 8)$ ,  $D(4, 6)$ . Prove that the area of the quadrilateral formed by joining in order, the middle points of the quadrilateral  $ABCD$  is equal to one half the area of  $ABCD$ .

11. Prove that the points  $P_1(x_1, y_1)$ ,  $P_2(x_2, y_2)$ ,  $P_3(x_3, y_3)$

$$\begin{array}{rcl} & x_1 & y_1 & 1 \\ \text{are collinear when and only when} & x_2 & y_2 & 1 \\ & x_3 & y_3 & 1 \end{array} = 0.$$

12. By making use of Ex. 11, prove that the points  $(1, 5)$ ,  $(-2, -16)$ ,  $(0, -2)$  are collinear.

#### 41. Pencil of lines.

(20) Given:  $A_1x + B_1y + C_1 = 0$ ,

(21)  $A_2x + B_2y + C_2 = 0$ ,

the equations of two straight lines. If these lines intersect, the equation

(22)  $(A_1x + B_1y + C_1) + k(A_2x + B_2y + C_2) = 0$

is the equation of a straight line through their point of intersection; because, 1. — it is an equation of the first degree and therefore is the equation of a straight line and 2. — it is satisfied by the coordinates of the point of intersection\* of the lines whose equations are (20) and (21). For different values of  $k$  (called the **parameter**) equation (22) is the equation of different lines of a system of straight lines, which consists of all\*\* lines

\* If the lines (20), (21) are parallel, line (22) is parallel to each of them. Why?

\*\* Line (20) is included in the system for the value  $k = 0$ ; line (21) is also regarded as belonging to the system, although it does not correspond to a value of  $k$ . If we replace (22) by  $k_1(A_1x + B_1y + C_1) + k_2(A_2x + B_2y + C_2) = 0$  this difficulty is avoided and the line whose equation is (21) is obtained by letting  $k_1 = 0$  and  $k_2 = 1$ .

through the intersection of the lines whose equations are (20) and (21) (or of a system of parallel lines). Such a system is called a pencil of lines. Some line of the pencil passes through any point we may wish to specify. For, if  $(x_1, y_1)$  be the coordinates of any such point not on the line whose equation is (21), the value of  $k$  is determined by substituting these coordinates in (22) and solving for  $k$ .

*Example 1.* Find the equation of the line through the point  $(2, -3)$  and the intersection of

$$2x + 3y - 5 = 0 \text{ and } 4x - 5y + 1 = 0.$$

*Solution:* The equation of the pencil of lines through the intersection of the two lines is

$$(2x + 3y - 5) + k(4x - 5y + 1) = 0.$$

We wish the line of the system which passes through the point  $(2, -3)$  and therefore the coordinates of this point must satisfy the equation.

$$\text{Hence} \quad -10 + k24 = 0,$$

$$\text{or} \quad k = \frac{5}{12}.$$

$$\text{Therefore } (2x + 3y - 5) + \frac{5}{12}(4x - 5y + 1) = 0.$$

$$\text{Simplifying, we have } 4x + y - 5 = 0.$$

*Example 2.* Find the equation of the line through the intersection of  $2x + 3y - 5 = 0$  and  $4x - 5y + 1 = 0$  and perpendicular to the line  $3x - y + 3 = 0$ .

*Solution:* The equation of the system of lines through the intersection of the two lines is

$$(2x + 3y - 5) + k(4x - 5y + 1) = 0,$$

$$\text{or} \quad (2 + 4k)x + (3 - 5k)y + (k - 5) = 0.$$

The slope of any line of the system is  $-\frac{2+4k}{3-5k}$

The slope of the line  $3x - y + 3 = 0$  is 3.

Since these lines are to be perpendicular, we have

$$\frac{2+4k}{3-5k} = \frac{1}{3},$$

or  $k = -\frac{5}{17}.$

$\therefore$  The equation of the desired line is

$$(2x + 3y - 5) - \frac{5}{17}(4x - 5y + 1) = 0,$$

or  $x + 3y - 4 = 0.$

The student may think that the last two examples could have been solved with less labor, by first finding the coordinates of the point of intersection of the given lines. In this particular case, this may be true, in view of the fact that these coordinates are simple numbers, viz. (1, 1). But the method described here often saves much time and furthermore it involves a valuable general principle of analytic geometry.

### Exercises

In each of the following examples find the equation of the line through the intersection of the given lines and satisfying the additional condition given. Use the method of § 41.

1.  $4x + 5y - 12 = 0$ ,  $3x - 2y - 6 = 0$ , and passing through the origin.
2.  $3x + 7y - 1 = 0$ ,  $6x + 2y - 8 = 0$ , and passing through the point (3, -1).

3.  $6x + 8y - 2 = 0$ ,  $4x - y - 3 = 0$ , and passing through the point  $(2, 1)$ .
4.  $x + 6y - 6 = 0$ ,  $5x - 2y - 10 = 0$ , and parallel to  $x - 2y + 4 = 0$ .
5.  $4x - y - 2 = 0$ ,  $4x + 3y - 2 = 0$ , and parallel to  $2x + 3y - 7 = 0$ .
6.  $x - y + 7 = 0$ ,  $2x - y + 3 = 0$ , and parallel to  $3x - y - 8 = 0$ .
7.  $5x - 2y + 8 = 0$ ,  $4x - 5y + 10 = 0$ , and perpendicular to  $x - 2y + 4 = 0$ .
8.  $4x - y - 2 = 0$ ,  $x + y + 3 = 0$ , and perpendicular to  $2x + y - 7 = 0$ .
9.  $x + y - 3 = 0$ ,  $x - y + 1 = 0$ , and perpendicular to the  $x$ -axis.
10.  $x - y + 4 = 0$ ,  $2x - 3y + 6 = 0$ , and having an  $x$ -intercept equal to 2.
11.  $2x + 3y - 7 = 0$ ,  $3x - y + 2 = 0$ , and having a  $y$ -intercept equal to 3.
12.  $2x - y - 7 = 0$ ,  $x = y - 3$ , and intersection of  $x + y + 7 = 0$ ,  $4x + 3y - 2 = 0$ .

**42. Two (or more) straight lines.** In § 26 we saw that an equation of the form

$$(23) \quad (A_1x + B_1y + C_1)(A_2x + B_2y + C_2) = 0$$

is satisfied by the coordinates  $(x, y)$  of every point on either of the straight lines whose equations are

$$(24) \quad A_1x + B_1y + C_1 = 0 \quad \text{and}$$

$$(25) \quad A_2x + B_2y + C_2 = 0,$$

and by no others, and that the locus of (23), therefore, consists of the two straight lines whose equations are (24) and (25).

An equation of the second degree in  $x$  and  $y$  may then have two straight lines as its locus. This will be the case when the right-hand member is zero, and the left-hand member can be factored into two *real* expressions of the first degree.

Thus a quadratic equation in  $x$  alone,

$$(26) \quad Ax^2 + Dx + C = 0,$$

will, in general, have for its locus two straight lines parallel to the  $y$ -axis whose equations are  $x = x_1$  and  $x = x_2$ , provided the roots  $x_1, x_2$  of the equation are real and distinct. For example, the equation

$$x^2 - 3x - 4 = 0$$

has for its locus the two parallel lines  $x = 4$  and  $x = -1$ . If  $x_1 = x_2$ , there is but one line, although the situation is frequently described by saying that the two lines coincide. If the roots are imaginary there is no locus.

Similarly, a quadratic equation in  $y$  alone will, in general, have for its locus two lines parallel to the  $x$ -axis, provided its roots are real and distinct.

A **homogeneous** quadratic equation in  $x$  and  $y$ , that is to say one in which every term is of the second degree in  $x$  and  $y$ ,

$$(27) \quad Ax^2 + Fxy + By^2 = 0,$$

can be written as an equation in  $\frac{y}{x}$  of the form

$$B \frac{y^2}{x^2} + F \cdot \frac{y}{x} + A = 0,$$

which is quadratic in terms of  $\frac{y}{x}$  if  $B \neq 0$ .

If this has two real distinct roots  $m_1$  and  $m_2$ , equation (27) has for its locus the two lines through the origin,

$$y = m_1x \text{ and } y = m_2x.$$

If  $B = 0$  equation (27) takes the form

$$x(Ax + Fy) = 0$$

which has for its locus the lines whose equations are  $x = 0$  and  $Ax + Fy = 0$ .

*Example.* Plot the locus of

$$y^2 - 5xy + 6x^2 = 0.$$

*Solution:* The given equation can be written as

$$(y - 3x)(y - 2x) = 0.$$

Hence the locus is the two lines whose equations are

$$y = 3x \text{ and } y = 2x.$$

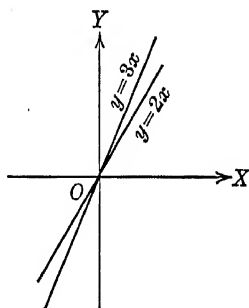


FIG. 54

### Exercises

Determine, when it exists, the locus of each of the following equations:

1.  $x^2 + x - 6 = 0$ .

2.  $x^2 + 4xy + 4y^2 = 0$ .

3.  $x^2 - 6x + 9 = 0$ .

4.  $y^2 - 3xy + 2x^2 = 0$ .

5.  $x^3 - 4x^2 - 5x = 0$ .

6.  $x^2 + 4x - 5 = 0$ .

7.  $x^2 + 4xy + y^2 = 0$ .

8.  $x^2 - x + 1 = 0$ .

9.  $x^2 - 2xy = 0$ .

10.  $xy = 0$ .

11. Prove that the equation  $Ax^2 + Fxy + By^2 = 0$  has for its locus two distinct straight lines if  $F^2 - 4AB > 0$ , two coincident straight lines if  $F^2 - 4AB = 0$ , and the origin only, if  $F^2 - 4AB < 0$ .
12. Prove that  $x^2 - xy - 3x - 2y^2 + 3y + 2 = 0$  represents two straight lines. [Hint: Solve the equation for  $y$ .]
13. Prove that  $6y^2 + y - x^2 + xy - 2x - 1 = 0$  represents two straight lines. Find the angle between them.

**43. Geometric theorems.** The methods of this chapter can readily be applied to the solution of geometric theorems. For example, prove that the medians of a triangle are concurrent.

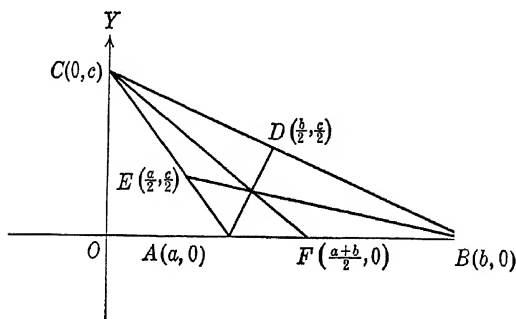


FIG. 55

**Solution:** We first choose the axes of coordinates so as to simplify the work. Given *any* triangle  $ABC$ , we can choose the  $x$ -axis to lie along one side of the triangle,  $AB$  say, and the  $y$ -axis to be the line through  $C$  perpendicular to  $AB$ . The coordinates of the vertices then assume the form  $A(a, 0)$ ;  $B(b, 0)$ ,  $C(0, c)$ . The coordi-

nates of the mid-points of the sides are then as indicated in the figure, namely,

$$D \left( \frac{b}{2}, \frac{c}{2} \right), \quad E \left( \frac{a}{2}, \frac{c}{2} \right), \quad F \left( \frac{a+b}{2}, 0 \right)$$

The equation of  $AD$  is

$$y - 0 = \frac{\frac{c}{2}}{\frac{a}{2} - a} (x - a),$$

which, when simplified, gives

$$AD: cx + (2a - b)y - ca = 0.$$

Similarly we find

$$BE: cx + (2b - a)y - cb = 0.$$

$$CF: 2cx + (a + b)y - (a + b)c = 0.$$

To find the point of intersection of  $AD$  and  $BE$ , we solve their equations simultaneously. Subtracting the left-hand members eliminates  $x$  and gives

$$3(a - b)y = c(a - b).$$

Since  $a \neq b$  (why?), we conclude that  $y = \frac{c}{3}$ . If this

value of  $y$  be substituted in either of the two equations, we get  $x = \frac{1}{3}(a + b)$ . The coordinates of the point of intersection of  $AD$  and  $BE$  are

$$\left( \frac{a + b}{3}, \frac{c}{3} \right).$$

To show that  $CF$  also passes through this point, we substitute these coordinates in the equation of  $CF$ . This gives

$$2c \frac{a+b}{3} + (a+b) \frac{c}{3} - (a+b)c = 0,$$

which is an identity. Our theorem is then proved.

We could have avoided the computation of the coordinates of the points of intersection, by observing that by adding the equations of  $AD$  and  $BE$ , we obtain the equation of  $CF$ . This proves that the three lines belong to the same pencil; and since they are not parallel they are concurrent.

### PROBLEMS

1. Prove the theorem of § 43 by using the choice of axes indicated in the adjoining figure,  $A(0, 0)$ ,  $B(a, 0)$ ,  $C(b, c)$ ,  $a \neq 0$ ,  $c \neq 0$ .

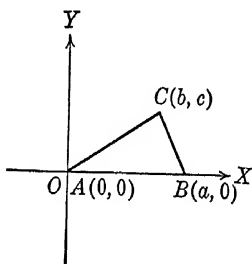


FIG. 56

2. Prove that the altitudes of any triangle meet in a point. Use the axes and notation of § 43; it will be found that the altitudes meet in the point

$$\left(0, -\frac{ab}{c}\right).$$

3. Prove that the perpendicular bisectors of the sides of any triangle meet in a point. Use the axes and notation of § 43;

it will be found that the coordinates of the desired point are

$$\left( \frac{a+b}{2}, \frac{ab+c^2}{2c} \right).$$

4. Prove that in any triangle the point of intersection of the medians, the point of intersection of the altitudes, and the point of intersection of the perpendicular bisectors of the sides lie on the same line. In what ratio does the first point divide the segment joining the other two? (Use the results of § 43 and Exs. 2 and 3.)
5. Prove that in a trapezoid the diagonals and the line joining the mid-points of the parallel sides are concurrent.
6. Prove that in any parallelogram  $ABCD$  the vertex  $D$ , the mid-point of  $AB$ , and a point of trisection of the diagonal  $AC$  are collinear.
7. Prove that in a trapezoid the non-parallel sides and the line joining the mid-points of the parallel sides are concurrent.
8. Prove that the altitudes on the legs of an isosceles triangle are equal.
9. Prove that the three altitudes of an equilateral triangle are equal.
10. Prove that the sum of the absolute distances of any point within an equilateral triangle from the sides of the triangle is equal to an altitude of the triangle.
11. If the equations of the sides of a triangle are

$$a_1x + b_1y + c_1 = 0,$$

$$a_2x + b_2y + c_2 = 0,$$

$$a_3x + b_3y + c_3 = 0,$$

prove that the area of the triangle is

$$\pm \frac{[a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)]^2}{2(a_1b_2 - a_2b_1)(a_2b_3 - a_3b_2)(a_3b_1 - a_1b_3)}.$$

12. Prove that in parallelogram  $ABCD$  the vertex  $D$ , the point which divides  $AB$  in the ratio  $1:m$ , and the point which divides  $AC$  in the ratio  $1:m+1$  are collinear.

### MISCELLANEOUS EXERCISES

- Find the equation of the line through  $(2, -3)$  parallel to the line through  $(2, 3)$ ,  $(-4, 1)$ .
- Find the coordinates of the vertices of the triangle the equations of whose sides are  $x+y=2$ ,  $2x-y=1$ ,  $3x+y=9$ .
- A diagonal of a square joins the points  $(2, 3)$  and  $(4, 6)$ . Find the coordinates of the other vertices.
- The base of an equilateral triangle is the segment joining  $(2, 0)$  to  $(6, 0)$ . Find the third vertex.
- A line has equal intercepts and passes through  $(3, 4)$ . Find its equation.
- Find the equation of the line perpendicular to  $2x-y=8$  and bisecting the segment joining  $(-2, 4)$  and  $(4, 6)$ .
- Find the distance from  $(5, -6)$  to the line whose intercepts are  $a=-2$ ,  $b=3$ .
- Given the triangle  $A(2, 0)$ ,  $B(6, 4)$ ,  $C(4, -6)$ :
  - Find the equation of  $AB$ .
  - Find the length of the altitude from  $C$ .
  - Find the length of  $AB$ .
  - Using (b) and (c) find the area of the triangle.
  - Check (d) using the formula for area (§ 40).

- f) Find the equations of the medians.  
 g) Prove that the medians are concurrent.  
 h) Find the equations of the altitudes.  
 i) Prove that the altitudes are concurrent.  
 j) Find the equations of the perpendicular bisectors of the sides.  
 k) Prove that the perpendicular bisectors of the sides are concurrent.
9. Same as Ex. 8 but for the triangle  $A(-2, 4)$ ,  $B(4, 6)$ ,  $C(6, -8)$ .
10. Prove that the acute angle between  $Ax + By + 1 = 0$  and  $(A + B)x - (A - B)y + K = 0$  is  $45^\circ$ .
11. Prove that the equation of the line through  $(x_1, y_1)$  parallel to  $Ax + By + C = 0$  is  $A(x - x_1) + B(y - y_1) = 0$ .
12. Prove that the equation of the line through  $(x_1, y_1)$  perpendicular to  $Ax + By + C = 0$  is  $B(x - x_1) - A(y - y_1) = 0$ .
13. Prove that the tangent of the angle from

$$A_1x + B_1y + C_1 = 0 \text{ to } A_2x + B_2y + C_2 = 0$$

is 
$$\frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}.$$

14. Find the slopes of the lines bisecting the angles formed by two lines whose slopes are 1 and 7.
15. Find the equation of the line through  $(2, 1)$  and forming in the first quadrant, with the coordinate axes, a triangle of area 4.
16. Prove that all lines  $\frac{x}{a} + \frac{y}{b} = 1$  pass through the point  $\left(\frac{1}{2}, \frac{1}{2}\right)$   
 if  $\frac{1}{a} + \frac{1}{b} = 2$ .

17. Three vertices of a parallelogram are  $(-6, 0)$ ,  $(4, 4)$ ,  $(2, 5)$ . Find the fourth vertex; three solutions.
18. Find the point on the line  $4x - 3y - 2 = 0$  which is equidistant from the points  $(4, 2)$  and  $(10, 4)$ .
19. Find  $k$  so that  $kx + 3y = 7$  will pass through the intersection of  $7x - 8y = 3$  and  $3y - 5 = 0$ .
20. Prove that the lines  $y = mx + b$  pass through a common point if  $m = b$ . What are the coordinates of this point?
21. The center of a circle is at  $(4, 3)$  and the circle is tangent to  $5x + 13y - 1 = 0$ . Find the radius of the circle.
22. The center of a circle lies on  $y - 2x = 0$ . If the circle passes through  $(4, 6)$ ,  $(10, 8)$ , find the coordinates of the center of the circle.
23. Find the equations of the two lines passing through  $(3, 0)$  such that perpendiculars drawn to them from  $(-6, -6)$  are of length 3.
24. Find the equation of the line through the intersection of  $y = -2x + 5$  and  $y = 7x - 4$  whose angle of inclination is  $45^\circ$ .
25. Find the center and radius of the circle circumscribed about the triangle  $(-2, -2)$ ,  $(4, 10)$ ,  $(6, 4)$ .
26. Find the equations of the sides of the square, if the coordinates of two opposite vertices are  $(4, 4)$ ,  $(10, 8)$ .
27. The vertex of the right angle of a right isosceles triangle is at  $(4, 3)$ . The opposite side lies on the line  $y = 3x + 6$ . Find the equations of the other two sides.
28. Given the equation  $Ax + By + C = 0$ . Find the relation among the coefficients so that:
  - a) the  $y$ -intercept is 7.

- b) the slope is 3.
  - c) the given line is parallel to  $2x - 5y = 7$ .
  - d) the line is parallel to the  $x$ -axis.
  - e) the line is parallel to the  $y$ -axis.
  - f) the line passes through  $(3, -2)$ .
29. Find the equations of the lines through the origin and through the points of trisections of the segment of the line  $2x + 3y = 12$  which is intercepted between the axes.
30. A perpendicular from the origin meets a line at  $(-2, 5)$ . Find the equation of the line.
31. For what values of  $k$  will the lines  $kx + y + 7 = 0$  and  $4x + ky + 3 = 0$  be parallel?

## CHAPTER V

### THE CIRCLE

**44. The center-radius equation.** Let the point  $P(x, y)$  lie on the circle with center  $C(h, k)$  and radius  $r$ .

Then  $CP = r$ ,

or

$$(1) \sqrt{(x-h)^2 + (y-k)^2} = r.$$

Hence

$$(2) (x-h)^2 + (y-k)^2 = r^2.$$

FIG. 57

Moreover, whenever the coordinates of a point  $P$  satisfy (2) they also satisfy (1) which is simply an equality between the positive square roots of the members of (2). Hence, if  $x$  and  $y$  are the coordinates of a point on the circle, equation (2) is satisfied. Moreover, it is not satisfied by the coordinates of a point not on the circle. Therefore, (2) is the equation of the circle. If the center is at the origin,  $h = k = 0$ , and the equation becomes

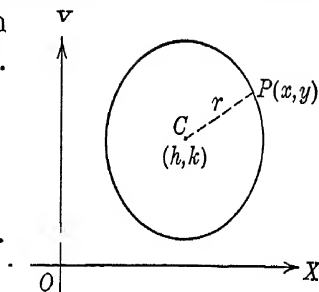
$$(2') \quad x^2 + y^2 = r^2.$$

*Example.* Find the equation of the circle with center at  $(-2, 1)$  and radius 3.

Solution:  $h = -2, k = 1, r = 3$ .

$\therefore$  The equation is  $(x+2)^2 + (y-1)^2 = 9$ ,

or 
$$x^2 + y^2 + 4x - 2y - 4 = 0.$$



## Exercises

Find the equations of the following circles:

1. Center at  $(3, 4)$ , radius = 5.
2. Center at  $(0, 0)$ , radius = 6.
3. Center at  $(4, -1)$ , radius = 3.
4. Center at  $(0, 0)$ , radius = 5.
5. Center at  $(a, -a)$ , radius =  $a$ .
6. Center at  $(a, a)$ , radius =  $a\sqrt{2}$ .
7. Diameter, the segment from  $(4, 2)$  to  $(10, 6)$ .
8. Diameter, the segment from  $(-4, 6)$  to  $(6, -2)$ .
9. Diameter, the segment from  $(0, 0)$  to  $(2a, 2a)$ .
10. Center at  $(3, 4)$  and touching the  $x$ -axis.
11. Center at  $(-2, 4)$  and touching the  $y$ -axis.
12. Center at  $(a, a)$  and touching both axes.
13. Touching the  $y$ -axis at the origin and radius = 2. (Two solutions.)
14. Touching the  $x$ -axis at  $(3, 0)$  and radius = 2. (Two solutions.)
15. Center at  $(2, 1)$  and tangent to  $3x + 4y = 1$ .
16. Center at  $(-2, 5)$  and tangent to  $5x - 12y = 3$ .
17. Center at  $(a, a)$  and tangent to  $ax - by = 1$ .
18. Center on  $y = x$  and tangent to the  $x$ -axis at  $(4, 0)$ .
19. Center on  $3y = x$  and tangent to  $y$ -axis at  $(0, 3)$ .
20. Center at the intersection of  $4x - y = 3$  and  $2x + y = 3$  and radius of 4.

**45. The general equation.** If the equation

$$(x - h)^2 + (y - k)^2 = r^2$$

is expanded it becomes

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0.$$

This is of the form

$$(3) \quad x^2 + y^2 + Dx + Ey + C = 0,$$

where  $D$ ,  $E$ , and  $C$  are constants. The question now arises, is every equation of the form (3) the equation of a circle?

Let us arrange the terms of (3) as follows:

$$x^2 + Dx + y^2 + Ey = -C$$

We complete the square of the first group of two terms by adding  $(\frac{1}{2}D)^2$ , and the second group by adding  $(\frac{1}{2}E)^2$ . This gives

$$x^2 + Dx + \frac{D^2}{4} + y^2 + Ey + \frac{E^2}{4} = -C + \frac{D^2}{4} + \frac{E^2}{4},$$

or

$$(4) \quad \left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = D^2 + E^2 - 4C$$

This has the form  $(x - h)^2 + (y - k)^2 = r^2$ , and has a circle for its locus with center at  $(-\frac{1}{2}D, -\frac{1}{2}E)$ , and radius  $\frac{1}{2}\sqrt{D^2 + E^2 - 4C}$ , *provided* the right-hand member of the equation is *positive*. If the right-hand member is negative, no (real) numbers  $x, y$  can satisfy the equation, since the sum of the squares of two real numbers cannot equal a negative number; in this case

the equation has no locus. If the right-hand member of (4) is zero, the point  $(-\frac{1}{2}D, -\frac{1}{2}E)$  is the only point whose coordinates satisfy the equation and therefore the locus is a single point.

The result of our discussion can be stated as follows:  
The locus of  $x^2 + y^2 + Dx + Ey + C = 0$  is

- a) A *circle*, if  $D^2 + E^2 - 4C > 0$ . Center at  $(-\frac{1}{2}D, -\frac{1}{2}E)$ , radius,  $\frac{1}{2}\sqrt{D^2 + E^2 - 4C}$ .
- b) A *point*, if  $D^2 + E^2 - 4C = 0$ . Point is  $(-\frac{1}{2}D, -\frac{1}{2}E)$ .
- c) No *locus*, if  $D^2 + E^2 - 4C < 0$ .

### Exercises

Determine what is the locus of each of the following equations. Use the method of completing the squares; do *not* merely substitute in the formulas derived in the text. Whenever the locus is a circle, find its center, radius, and draw its graph.

1.  $x^2 + y^2 - 6x + 8y = 0$ .
2.  $x^2 + y^2 + 4x - 2y + 5 = 0$ .
3.  $x^2 + y^2 - 2x - 6y + 17 = 0$ .
4.  $x^2 + y^2 + 10x - 24y = 0$ .
5.  $x^2 + y^2 - 4x - 12 = 0$ .
6.  $x^2 + y^2 - 6x - 7 = 0$ .
7.  $x^2 + y^2 + 1 = 0$ .
8.  $x^2 + y^2 - 3x - 5y - \frac{1}{2} = 0$ .
9.  $2x^2 + 2y^2 - 4x - 8y + 5 = 0$ .

10.  $3x^2 + 3y^2 + 4x - 2y + 7 = 0$ .

11.  $Ax^2 + Ay^2 + Dx + Ey + C = 0$ . Discuss completely.

Find the equation of the circle which satisfies each of the following conditions:

12. Center at  $(1, 2)$  and passing through the point  $(4, 6)$ .

13. Center at  $(-2, 3)$  and passing through the point  $(1, -2)$ .

14. Center at  $(-1, 4)$  and passing through the origin.

15. Center at  $(0, 2)$  and tangent to the  $x$ -axis.

16. Center on the line  $x = 3$  and tangent to both axes.

17. Center at  $(1, -2)$  and tangent to the line  $x + y - 2 = 0$ .

18. Passing through the points  $(1, -2)$  and  $(3, 2)$  and with center on the  $y$ -axis.

19. Passing through the points  $(2, 1)$  and  $(4, 3)$  and with center on the line  $x - y + 1 = 0$ .

**46. The circle through three points.** Three points, if they do not lie on the same straight line, determine one and only one circle. We note that the center-radius equation (2), and the general equation (3), both contain three arbitrary constants,  $h, k, r$  in the former case and  $D, E, C$  in the latter. The requirement that the coordinates of a given point satisfy either of these equations gives an equation connecting the constants. For example, suppose the coordinates of the three points to be  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$ , we then have, using equation (3), the three equations

$$x_1^2 + y_1^2 + Dx_1 + Ey_1 + C = 0,$$

$$x_2^2 + y_2^2 + Dx_2 + Ey_2 + C = 0,$$

$$x_3^2 + y_3^2 + Dx_3 + Ey_3 + C = 0,$$

from which the values of the constants  $D$ ,  $E$ , and  $C$  can be found, provided, of course, that the three points are not collinear.

If we consider the expression  $x^2 + y^2$  as  $(x^2 + y^2) \cdot 1$ , we can then consider the last three equations and equation (3) as a system of homogeneous linear equations in the quantities, 1,  $D$ ,  $E$ , and  $C$ . If we eliminate these quantities by § 3, we have as the equation of the circle,

$$(5) \quad \begin{vmatrix} (x^2 + y^2) & x & y & 1 \\ (x_1^2 + y_1^2) & x_1 & y_1 & 1 \\ (x_2^2 + y_2^2) & x_2 & y_2 & 1 \\ (x_3^2 + y_3^2) & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

If this determinant is expanded, the coefficient of  $x^2 + y^2$  is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Since the three points are not collinear, the value of this determinant is not zero. If the three points are collinear, the value of this determinant is zero and the original determinant will give the equation of the straight line through the three given points.

*Example.* Find the equation of the circle through the points  $P_1 (2, 2)$ ,  $P_2 (-5, 1)$ ,  $P_3 (4, -2)$ .

Solution: The general equation of a circle is

$$x^2 + y^2 + Dx + Ey + C = 0.$$

Since by hypothesis this circle is to pass through the points  $P_1$ ,  $P_2$ , and  $P_3$ , the coordinates of these points

must satisfy the equation. This gives the three conditions:

$$\begin{cases} 4 + 4 + 2D + 2E + C = 0, \\ 25 + 1 - 5D + E + C = 0, \\ 16 + 4 + 4D - 2E + C = 0, \end{cases}$$

or

$$\begin{cases} 2D + 2E + C = -8, \\ 5D - E - C = 26, \\ 4D - 2E + C = -20. \end{cases}$$

Solving these equations, we have

$$D = 2, \quad E = 4, \quad C = -20.$$

The desired equation is then

$$x^2 + y^2 + 2x + 4y - 20 = 0.$$

This problem could also have been solved as follows. The equations of the perpendicular bisectors of the chords  $P_1P_2$  and  $P_2P_3$  are  $7x + y = -9$  and  $3x - y = -1$  respectively. These lines meet in the point  $C(-1, -2)$  which is the center of the required circle. The radius of the required circle is the distance  $CP_1$  or 5. Therefore the required equation is  $(x+1)^2 + (y+2)^2 = 25$  or  $x^2 + y^2 + 2x + 4y - 20 = 0$ .

If we use determinants the equation can be written as

$$\begin{vmatrix} (x^2 + y^2) & x & y & 1 \\ 8 & 2 & 2 & 1 \\ 26 & -5 & 1 & 1 \\ 20 & 4 & -2 & 1 \end{vmatrix} = 0,$$

which expanded gives  $x^2 + y^2 + 2x + 4y - 20 = 0$ .

## Exercises

Find the equations of the circles through the following sets of three points. Draw the corresponding figures:

1.  $(1, 0), (0, -1), (0, 0)$ .      2.  $(1, 0), (-1, -2), (1, -2)$ .

3.  $(1, 6), (2, 5), (-6, -1)$ .      4.  $(1, 3), (1, 1), (-2, 3)$ .

5. Find the equation of the circle circumscribing the triangle formed by the coordinate axes and the line  $2x + 3y = 6$ .

6. Find the equation of the circle circumscribing the triangle the equations of whose sides are

$$x + y - 1 = 0, \quad x - y + 2 = 0, \quad 2x + y + 3 = 0.$$

7. Determine if the points  $(2, 0), (0, 4), (2, 2), (1, 1)$  lie on a circle.

**47. A pencil of circles.** Given the equations of two circles:

$$(6) \quad x^2 + y^2 + D_1x + E_1y + C_1 = 0 \text{ and}$$

$$(7) \quad x^2 + y^2 + D_2x + E_2y + C_2 = 0.$$

If the second equation be multiplied by any constant  $k$  and added to the first equation, we obtain

$$(8) \quad x^2 + y^2 + D_1x + E_1y + C_1 + k(x^2 + y^2 + D_2x + E_2y + C_2) = 0,$$

or

$$(9) \quad (1 + k)x^2 + (1 + k)y^2 + (D_1 + kD_2)x + (E_1 + kE_2)y + C_1 + kC_2 = 0.$$

Equation (9) represents in general the equation of a circle for every value of  $k$  except  $k = -1$ , and is called

a **pencil of circles**. A general discussion of the properties of the pencil of circles is beyond the scope of this text. If the two circles intersect in two points and  $k \neq -1$ , then locus (9) is a pencil of circles passing through the two points of intersection; if  $k = -1$ , equation (9) is the equation of the common chord.

*Example 1.* Find the equation of the common chord of the circles

$$x^2 + y^2 + 3x - 5y = 0,$$

$$x^2 + y^2 + x - 4y + 1 = 0.$$

Solution: Eliminating  $x^2$  and  $y^2$ , we have the equation of the common chord.

$$\therefore 2x - y - 1 = 0.$$

*Example 2.* Find the coordinates of the points of intersection of

$$x^2 + y^2 + 3x - 5y = 0 \text{ and } x^2 + y^2 + x - 4y + 1 = 0.$$

Solution: The equation of the common chord is

$$2x - y - 1 = 0.$$

Solving for  $y$ , we have  $y = 2x - 1$ . Substituting in the first equation, we have

$$x^2 + (2x - 1)^2 + 3x - 5(2x - 1) = 0$$

$$\text{or} \quad 5x^2 - 11x + 6 = 0.$$

$$\text{Hence, } (x - 1)(5x - 6) = 0 \text{ or } x = 1, x = \frac{6}{5}.$$

Substituting these values of  $x$  in the equation of the

first degree, we find the coordinates of the points of intersection to be  $(1, 1)$ ,  $(\frac{6}{5}, \frac{7}{5})$ .

*Example 3.* Find the equation of the circle through the intersection of the circles whose equations are

$$x^2 + y^2 + 3x - 5y = 0$$

and  $x^2 + y^2 + x - 4y + 1 = 0,$

and the point  $(-1, 2)$ .

Solution: The equation of the pencil of circles through the points of intersection of the given circles is

$$(x^2 + y^2 + 3x - 5y) + k(x^2 + y^2 + x - 4y + 1) = 0.$$

We desire the circle of the pencil that passes through  $(-1, 2)$ .

$$\therefore (1 + 4 - 3 - 10) + k(1 + 4 - 1 - 8 + 1) = 0,$$

or  $k = -\frac{8}{3}.$

Hence,

$$(x^2 + y^2 + 3x - 5y) - \frac{8}{3}(x^2 + y^2 + x - 4y + 1) = 0,$$

or simplifying we have  $5x^2 + 5y^2 - x - 17y + 8 = 0$   
as the equation of the desired circle.

### Exercises

Find the equation of the common chord in each of the following examples:

1.  $x^2 + y^2 = 4$ ,  $x^2 + y^2 - 2y = 0$ .

2.  $x^2 + 2x + y^2 + 7y - 1 = 0$ ,  $x^2 + y^2 + 4y - 2 = 0$ .

3.  $x^2 - 4x + y^2 - 6y - 2 = 0$ ,  $x^2 - 6x + y^2 - 2y - 3 = 0$ .

Find the coordinates of the points of intersection of each of the following pairs of loci:

$$\begin{aligned} 4. \quad & x^2 + y^2 - 4x + 6y + 3 = 0, \\ & 3x - y - 3 = 0. \end{aligned}$$

$$\begin{aligned} 5. \quad & x^2 + y^2 - 2x - 4y - 12 = 0, \\ & x - 2y + 5 = 0. \end{aligned}$$

$$\begin{aligned} 6. \quad & x^2 + y^2 - 4y - 6 = 0, \\ & x^2 + y^2 + 2x - 4 = 0. \end{aligned}$$

$$\begin{aligned} 7. \quad & x^2 + y^2 + 3x - 6y - 1 = 0, \\ & x^2 + y^2 = 13. \end{aligned}$$

$$\begin{aligned} 8. \quad & x^2 + y^2 = 17, \\ & 4x + y = 17. \text{ Interpret the result.} \end{aligned}$$

$$\begin{aligned} 9. \quad & x^2 + y^2 - 6x - 12y + 25 = 0, \\ & x^2 + y^2 = 5. \text{ Interpret the result.} \end{aligned}$$

10. Find the equation of the circle through the intersections of  $x^2 + y^2 + x - 2y - 1 = 0$  and  $x^2 + y^2 + 2x - 4 = 0$  and (a) the origin; (b) the point (0, 3); (c) with its center on the line  $x = 2$ .

11. Find the equation of the circle through the intersections of  $x^2 + y^2 = 5$  and  $2x + y - 4 = 0$  and the origin.  
(Note: the circle can be obtained from

$$x^2 + y^2 - 5 + k(2x + y - 4) = 0. \text{ Why?})$$

12. What is the difference between the following pencils of circles?

$$a) (x^2 + y^2 + D_1x + E_1y + F_1) + k(x^2 + y^2 + D_2x + E_2y + F_2) = 0,$$

$$b) k(x^2 + y^2 + D_1x + E_1y + F_1) + (x^2 + y^2 + D_2x + E_2y + F_2) = 0,$$

$$c) k_1(x^2 + y^2 + D_1x + E_1y + F_1) + k_2(x^2 + y^2 + D_2x + E_2y + F_2) = 0.$$

**48. Tangents. Point-of-contact form.** Let  $P_1(x_1, y_1)$  be the point where the tangent touches the circle  $x^2 + y^2 = r^2$ . We wish to find the equation of the tangent at  $P_1$ . We know that the tangent is perpendicular to the radius  $OP_1$ . Since the slope of the radius is  $\frac{y_1}{x_1}$ , the

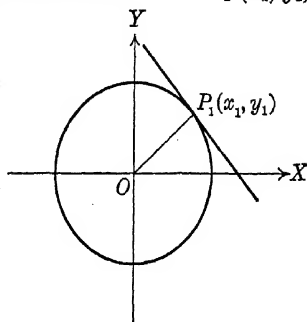


FIG. 58

slope of the tangent is  $-\frac{x_1}{y_1}$ .

The equation of the tangent is, therefore,

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1),$$

or  $x_1x + y_1y = x_1^2 + y_1^2 = r^2$ ,

the latter equality being due to the fact that the point  $(x_1, y_1)$  is on the circle and consequently  $x_1^2 + y_1^2 = r^2$ .

Hence we have:

The equation of the tangent to the circle  $x^2 + y^2 = r^2$  at the point  $(x_1, y_1)$  is

$$(10) \quad x_1x + y_1y = r^2.$$

It is left as an exercise for the student to show that by the same method, the equation of the tangent to the circle  $(x - h)^2 + (y - k)^2 = r^2$  at  $(x_1, y_1)$  can be written in the form

$$(x_1 - h)(x - h) + (y_1 - k)(y - k) = r^2.$$

If the equation of the circle is

$$x^2 + y^2 + Dx + Ey + C = 0,$$

we know from § 45 that

$$h = -\frac{D}{2}, \quad k = -\frac{E}{2}, \quad r = \frac{1}{2} \sqrt{D^2 + E^2 - 4C}.$$

∴ The equation of the tangent at  $P_1(x_1, y_1)$  is

$$\left(x_1 + \frac{D}{2}\right)\left(x + \frac{D}{2}\right) + \left(y_1 + \frac{E}{2}\right)\left(y + \frac{E}{2}\right) - \frac{D^2 + E^2 - 4C}{4},$$

or simplified,

$$(11) \quad x_1x + y_1y + \frac{D}{2}\left(\frac{x+x_1}{2}\right) + \frac{E}{2}\left(\frac{y+y_1}{2}\right) + C = 0.$$

### Exercises

Find the equation of the tangent to each of the following circles at the point indicated. In each case verify that the given point is on the circle.

1.  $x^2 + y^2 = 25$  at  $(4, -3)$ .      2.  $x^2 + y^2 = 13$  at  $(-2, -3)$
3.  $x^2 + y^2 = 49$  at  $(0, 7)$ .      4.  $x^2 + y^2 = 25$  at  $(3, 4)$ .
5.  $x^2 + y^2 = 41$  at  $(5, 4)$ .
6.  $x^2 + y^2 + 4x + 6y + 3 = 0$  at  $(-5, -4)$ .
7.  $2x^2 + 2y^2 - 5x + 3y = 0$  at  $(0, 0)$ .
8.  $x^2 + y^2 - 2x + 4y = 20$  at  $(4, 2)$ .
9.  $(x-1)^2 + (y+2)^2 = 20$  at  $(3, 2)$ .
10.  $(x-a)^2 + (y-b)^2 = a^2 + b^2$  at  $(2a, 2b)$ .

## MISCELLANEOUS EXERCISES

Find the coordinates of the center and the length of the radius of each of the following circles:

1.  $x^2 + y^2 - 8x + 6y - 2 = 0$ .
2.  $2x^2 + 2y^2 + 5x - 3y - 1 = 0$ .
3.  $x^2 + 2(a+b)x + y^2 + 2(a-b)y = 4ab$ .
4.  $x^2 + y^2 - ax = k$ .

Find the equation of the circle through:

5. (2, 0), (0, 2), (2, 4).
6. (5, 1), (2, 3), (0, 1).
7. (5, 7), (1, 3), (3, 2).

Find the equation of the circle:

8. Center at (3, -2), tangent to  $3x + 4y + 4 = 0$ .
9. Center at (-6, 5), tangent to  $5x - 12y - 3 = 0$ .
10. Center on the  $y$ -axis and passing through (4, 6) and (6, 10).
11. Center on the  $x$ -axis and passing through (6, 4) and (8, -4).
12. Whose diameter is the segment (-2, 3), (5, -2).
13. Passing through (6, 2), (2, 0) and center on  $2x - y + 5 = 0$ .
14. Passing through (2, 1) and tangent to the coordinate axes. Two solutions.
15. Passing through (2, 2) and (4, 2) with radius 3. Two solutions.
16. Prove that the points (6, 2), (-1, -5), (0, 2) and (3, 3) lie on a circle.

17. Prove that the points  $(0, 0)$ ,  $(0, 4)$ ,  $(6, 0)$  and  $(5, -1)$  lie on a circle.
18. Find the equations of the two circles with centers on  $3y = 5x - 8$  and tangent to the coordinate axes.
19. Find the equation of the circle passing through the origin and having  $x$ -intercept  $a$ ,  $y$ -intercept  $-b$ .
20. Find the equation of the circle inscribed in the triangle formed by the coordinate axes and the line  $3x + 4y = 12$ .

**The angle between two curves. Orthogonality.**

If two curves meet at a point  $P$ , the acute angle between the tangents to the curves at  $P$  is called the angle between the curves at  $P$ . If the tangents are perpendicular to each other, the curves are said to be **orthogonal**.

21. Prove that if the circles

$$x^2 + y^2 + D_1x + E_1y + C_1 = 0,$$

$$x^2 + y^2 + D_2x + E_2y + C_2 = 0,$$

intersect orthogonally, the sum of the squares of the radii is equal to the square of the distance between the centers and

$$D_1D_2 + E_1E_2 = 2C_1 + 2C_2.$$

22. Prove that the circles

$$2x^2 + 2y^2 + 4x - 6y = 19,$$

$$x^2 + y^2 - 4x + 5y = 2,$$

are orthogonal.

23. Find the relation among the constants in order that

$Ax + By + C = 0$  be orthogonal to

$$x^2 + y^2 + Dx + Ey + F = 0.$$

24. Is there a circle orthogonal to each of the following circles?

$$x^2 + y^2 = 6,$$

$$x^2 + y^2 - 4y = 0,$$

$$x^2 + y^2 + 6x = 0$$

### GEOMETRIC PROBLEMS

25. On  $CA$ , the radius of a circle with center  $C$ , a circle is constructed with  $CA$  as a diameter. Prove analytically that any chord of the given circle drawn through  $A$  is bisected by the second circle.
26. Prove analytically that every angle inscribed in a semi-circle is a right angle.
27. Prove analytically that the circle through the mid-points of the sides of a triangle passes through the feet of the altitudes and also through the points halfway between the vertices and the point of intersection of the altitudes. This circle is called the nine point circle.
28. If the coordinates  $(x_1, y_1)$  of an external point  $P_1$  be substituted for  $x, y$  in the equation  $x^2 + y^2 + Dx + Ey + C = 0$ , prove that  $x_1^2 + y_1^2 + Dx_1 + Ey_1 + C$  is the square of the length of the segment of the tangent drawn to the circle from  $P_1$ .
29. Prove analytically that if a perpendicular  $PA$  is drawn from a point  $P$  of a circle to a diameter  $BC$ ,  $\overline{PA}^2 = BA \cdot AC$ .
30. Prove analytically that a radius perpendicular to a chord of a circle bisects the chord.
31. Prove analytically that the mid-point of an arc of a circle is equidistant from the chord of the arc and a tangent drawn at one end of the arc.

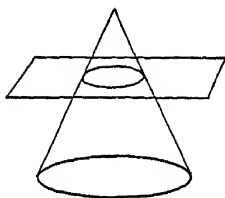
## LOCUS PROBLEMS

32. Find the locus of a point such that the sum of the squares of its distances from two fixed points is constant.
33. Find the locus of a point which moves so that the sum of the squares of its distances from the sides of a given square is constant.
34. A line is drawn through each of two fixed points  $A$  and  $B$  forming at  $P$  a constant angle  $\theta$ . Find the equation of the locus of  $P$ .
35. A point moves so that its distance from a fixed point  $A$  is always equal to  $k$  times its distance from another fixed point  $B$ . Show that the path generated is a circle if  $k \neq 1$ .
36. A line rotating about a fixed point  $O$  meets a fixed line in a point  $Q$ . Find the locus of a point  $P$  on  $OQ$  such that  $OQ \cdot OP$  is constant.
37. Find the locus of a point such that the square of its distance from the base of an isosceles triangle is equal to the product of its distances from the other two sides.
38. Find the locus of a point such that the length of a tangent drawn from it to one of two given circles is  $k$  times the length of a tangent drawn from it to the other circle.

## CHAPTER VI

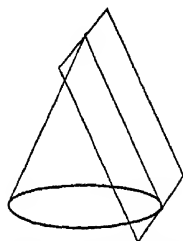
### CONICS

**49. A conic section or conic** is any plane section of a right circular cone.\* It is evident that the shape of a conic will depend upon the position of the cutting plane. For example, if the plane is parallel to the base of the cone and does not pass through the vertex of the cone,



Circle

FIG. 59



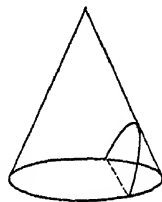
Two coincident lines

FIG. 60

we have a circle. If the plane passes through the vertex, we obtain either a point, two distinct straight lines, or one line. In the last case, the plane is tangent to the cone and it is frequently said that the section is two coincident lines.

If the cutting plane is parallel to a tangent plane, the section is called a **parabola**.

It should be noted that the cutting plane can intersect only one nappe of the cone and that the curve is open and contains but one branch.

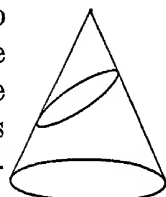


Parabola

FIG. 61

\* See Morgan, Foberg, Breckenridge, *Solid Geometry*, pages 579-583.

If the cutting plane is not parallel to an element and cuts only one nappe, the section is called an **ellipse**. It should be noted that the curve is closed. The orbits of the earth and other planets are approximately ellipses.

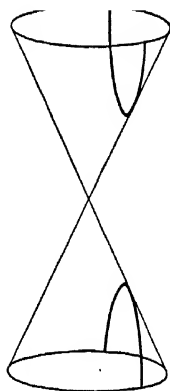


Ellipse  
FIG. 62

If the cutting plane intersects both nappes, and does not pass through the vertex, the section is called a **hyperbola**. This curve contains two branches both of which are open.

If the vertex of the cone recedes indefinitely, the cone approaches a cylinder and the section may be

- 1) a circle,
- 2) an ellipse,
- 3) two parallel lines,
- 4) two coincident lines.



Hyperbola  
FIG. 63

It was from the above point of view that the Greek mathematicians, especially Apollonius, 200 B.C., first studied the properties of these curves. In the present chapter we shall study the parabola, the ellipse, and the hyperbola by analytic methods. It can be shown that the definitions which follow lead to the curves just defined geometrically.\*

\* See Young and Morgan, *Mathematical Analysis*, pages 370-375.

**50. A parabola** is the locus of a point in a plane equidistant from a fixed point  $F$  in the plane called the **focus**, and a fixed line  $D$  in the plane (not passing through the focus), called the **directrix**. Hence in Fig. 64,  $PF = PD$ . The ratio  $PF : PD$  is called the **eccentricity**. Since these lengths are equal, the eccentricity of a parabola is always **one**.

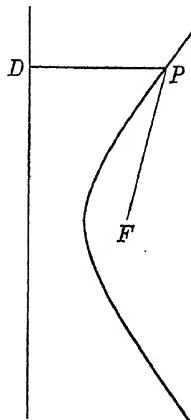


FIG. 64

**51. Construction of a parabola.** If a line be drawn through the focus perpendicular to the directrix meeting the latter in  $D$ , the mid-point  $V$  of  $DF$  is a point of the parabola since it is equidistant from  $D$  and  $F$ . It is called the **vertex** of the parabola. Any number of other points of the parabola can be constructed as follows. Draw a line  $l$  parallel to the directrix meeting the line through  $D$  and  $F$  in a point  $M$  and then with  $F$  as a center and  $DM$  as a radius describe an arc cutting  $l$  in  $P$  and  $P'$ . The points  $P, P'$  are points of the parabola (why?). By drawing other

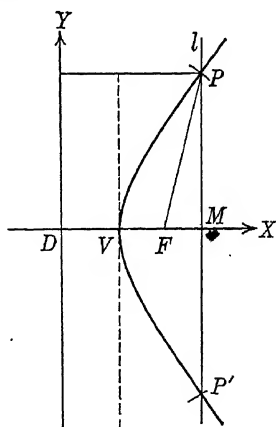


FIG. 65

lines  $l$ , any number of additional points of the parabola can be found. This method of construction shows that the parabola is symmetric with respect to the line  $DF$ , which is called the **axis** of the parabola; the line through the vertex  $V$  parallel to the directrix meets the curve in point  $V$  only and is a **tangent** to the curve. Moreover, we see that the parabola lies entirely on that side of this line on which the focus lies.

### Exercises

1. Construct a parabola given the directrix and the vertex.
2. Construct a parabola given the focus and the vertex.
3. Construct a parabola given the axis, focus, and a point on the curve.

#### 52. The equation of the parabola: Vertex at origin.

The form of the equation of a curve depends upon where we take the axes. The equation may be much simpler for one choice of axes than for others, and if the equation is to be useful we wish it to be as simple as possible. The equation of the parabola is particularly simple if the line joining the vertex to the focus  $F$  is taken as the  $x$ -axis and the vertex is taken as the origin  $O$ . If we designate the distance  $OF$  by  $p$ , the coordinates of  $F$  are  $(p, 0)$ .

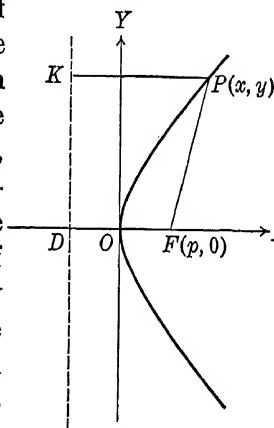


FIG. 66

The directrix is now

parallel to the  $y$ -axis and  $p$  units from it on the other side from the focus, so that its equation is

$$x + p = 0.$$

If  $P(x, y)$  is any variable point on the curve and the line through  $P$  parallel to the  $x$ -axis meets the directrix in  $K$  the definition of the curve states that

$$KP = FP.$$

But  $KP = x + p$ , and  $FP = \sqrt{(x - p)^2 + y^2}$ . Equating these values, we obtain

$$(1) \quad x + p = \sqrt{(x - p)^2 + y^2},$$

which is true when, and only when, the point  $(x, y)$  is on the parabola. The coordinates of a point which satisfy (1) also satisfy

$$(2) \quad (x + p)^2 = (x - p)^2 + y^2,$$

which reduces to

$$(3) \quad y^2 = 4px.$$

Conversely, any point  $(x, y)$  which satisfies (3) must satisfy (1) and hence must lie on the parabola. For, (3) is equivalent to (2) and from (2) we conclude that either

$$(4) \quad \sqrt{(x - p)^2 + y^2} = x + p,$$

or

$$(5) \quad \sqrt{(x - p)^2 + y^2} = -(x + p).$$

If  $p > 0$ , (3) shows that, in order for  $y$  to be a real num-

ber, we must have  $x \geq 0$ , and, in order to have a point in the plane,  $y$  must be a real number, but if (5) were true for  $x \geq 0$  we would have a positive number equal to a negative number which is impossible; hence (5) is impossible and (4) is equivalent to (1).

Hence, the equation of the parabola with focus at  $(p, 0)$  and vertex at the origin is

$$(3) \quad y^2 = 4px.$$

The equation of the directrix is

$$x = -p.$$

It is left as an exercise for the student to show that if the vertex be taken at  $(0, 0)$  and the focus at  $(-p, 0)$ , the equation of the parabola is

$$(6) \quad y^2 = -4px,$$

and the equation of the directrix is  $x = p$ . Fig. 67.

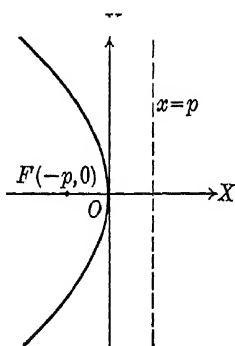


FIG. 67

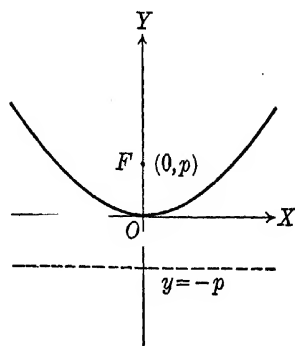


FIG. 68

If the vertex be taken at  $(0, 0)$  and the focus at  $(0, p)$ , the equation of the parabola is

$$(7) \quad x^2 = 4py,$$

and the equation of the directrix

is  $y = -p$ . Fig. 68.

If the vertex be taken at  $(0, 0)$  and the focus at  $(0, -p)$ , the equation of the parabola is

$$(8) \quad x^2 = -4py,$$

and the equation of the directrix

is  $y = p$ . Fig. 69.

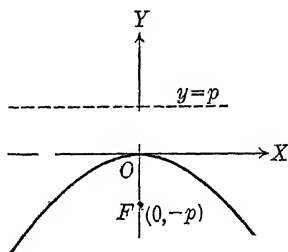


FIG. 69

**53. The latus rectum.** The chord of a parabola drawn through the focus perpendicular to the axis of the curve is called the **latus rectum** of the parabola; the length of this chord is also called by the same name. If the equation of the curve is taken as  $y^2 = 4px$ , the focus is at  $(p, 0)$ , the equation of the line through the focus perpendicular to the axis of the curve (the  $x$ -axis in this case) is  $x = p$ . This line cuts the curve in points for which  $y^2 = 4p \cdot p = 4p^2$ , i.e., for which  $y = \pm 2p$ . The length of the latus rectum of the parabola  $y^2 = 4px$  is  $4p$ , i.e., the length of the latus rectum of a parabola is four times the distance from the vertex to the focus, or twice the distance of the focus from the directrix.

### Exercises

For each of the following parabolas, find the coordinates of the vertex and the focus, the equation of the directrix, and the length of the latus rectum. Sketch the curve.

1.  $y^2 = 4x$ .

2.  $y^2 = -4x$ .

3.  $x^2 = 4y$ .

4.  $x^2 = -4y$ .      5.  $y^2 = 16x$ .      6.  $y^2 = -16x$ .  
 7.  $x^2 = 16y$ .      8.  $x^2 = -16y$ .      9.  $3y^2 - 7x = 0$ .  
 10.  $3y^2 + 7x = 0$ .    11.  $3x^2 - 7y = 0$ .    12.  $3x^2 + 7y = 0$ .

Derive the equations of the following parabolas using the definition of a parabola and not by substituting in a formula:

13. Vertex at origin, focus at  $(3, 0)$ .  
 14. Vertex at origin, focus at  $(-3, 0)$ .  
 15. Vertex at origin, focus at  $(0, 5)$ .  
 16. Vertex at origin, focus at  $(0, -5)$ .  
 17. By means of the method given in § 51, draw the parabola with focus 4 units from the directrix.  
 18. In the adjacent figure, right triangle  $ABC$  is so arranged that side  $BC$  will slide along the  $y$ -axis. A string of length  $CA$  has one end fastened at  $A$  and the other end at the point  $F$ . A pencil is pressed against the side  $CA$  of the triangle holding the string taut. Show that the pencil will describe a parabola as the triangle slides up along the  $y$ -axis.

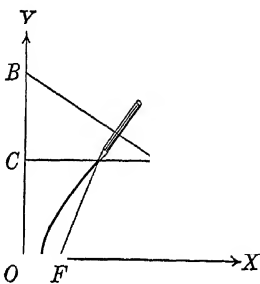


FIG. 70

19. If  $P_1(x_1, y_1)$  is any point on the parabola  $y^2 = 4px$ , prove that  $P_1F = x_1 + p$ . This distance is called the **focal radius** of  $P_1$ .

Find the equations of the following parabolas:

20. Vertex at  $(0, 0)$ , focus at  $(4, 0)$ .

21. Vertex at  $(0, 0)$ , directrix  $3y + 4 = 0$ .  
22. Vertex at  $(0, 0)$ , directrix  $2y - 3 = 0$ .  
23. Vertex at  $(4, 0)$ , directrix the  $y$ -axis.  
24. Vertex at  $(0, -4)$ , directrix the  $x$ -axis.  
25. If the  $x$ -axis is chosen as in § 52 but the  $y$ -axis coincides with the directrix, prove that the equation of the parabola is

$$y^2 = 4px - 4p^2.$$

26. If the  $x$ -axis is chosen as in § 52 but the  $y$ -axis is taken through the focus, prove that the equation of the parabola is

$$y^2 = 4px + 4p^2.$$

27. Find the equation of the parabola, vertex at the origin, axis coinciding with the  $x$ -axis, and passing through the point  $(3, 5)$ .  
28. Find the equation of the parabola, vertex at the origin, axis of curve coinciding with the  $y$ -axis, and passing through the point  $(-2, -3)$ .  
29. A parabola opens out along the negative half of the  $x$ -axis. Its focus is at  $(-3, 0)$  and its latus rectum is 12. Find the equation of the parabola.  
30. Prove that every parabola whose axis is parallel to the  $y$ -axis has an equation of the form

$$y = ax^2 + bx + c,$$

where  $a$ ,  $b$ , and  $c$  are constants and  $a \neq 0$ . [*Hint*: Find the equation of the parabola whose directrix is  $y = k$  and whose focus is  $(m, k + 2p)$  or  $(m, k - 2p)$ .]

31. Find the equation of the parabola, whose axis is parallel to the  $y$ -axis and which passes through the points  $(0, 0)$   $(1, 1)$   $(2, 4)$ . (See Ex. 30.)

Find the coordinates of the points of intersection of:

32.  $y^2 = 4x$ ,  $y - x - 1 = 0$ .

33.  $x^2 = 16y$ ,  $3y - 2x + 5 = 0$ .

34.  $y^2 = 8x$ ,  $y = x + 2$ .

35.  $x^2 - 8y = 0$ ,  $2x - y - 8 = 0$ .

54. An ellipse is the locus of a point in a plane such that the sum of its distances from two fixed points in the plane is a constant greater than the distance between the two fixed points. Each of the fixed points is called a **focus** (plural, *foci*).

If the foci are  $F_1$ ,  $F_2$  and  $P$  is any point on the ellipse, the definition requires that  $F_1P + F_2P = \text{constant}$  greater than  $F_2F_1$ .

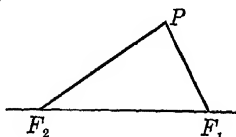


FIG. 71

The general shape of the curve is readily determined.

In fact, a simple mechanical device enables us to draw it.

Let two tacks be stuck in the paper at points  $F_1$  and  $F_2$  and the ends of a string tied to them. Draw the string tight with a pencil point as indicated in the figure. If the pencil be made to move on the paper,

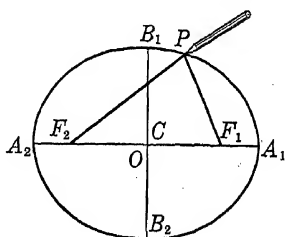


FIG. 71a

keeping the string tight, it will describe a curve for which  $F_1P + F_2P$  is constant, namely, the length of the string; it will therefore, by definition, describe an ellipse.

We immediately conclude from the method of construction that the ellipse is a smooth loop which is

symmetrical with respect to the line  $F_2F_1$  joining the foci, and also with respect to the perpendicular bisector of  $F_2F_1$ . The curve cuts  $F_2F_1$  in two points  $A_1$  and  $A_2$ , called **vertices**, and it cuts the other axis of symmetry in two points  $B_1$  and  $B_2$ .

The undirected segment  $A_2A_1$  is called the **major axis** of the ellipse, the undirected segment  $B_2B_1$  the **minor axis**, and the intersection of the two axes is called the **center**.

Let  $F_2F_1 = 2c$  and  $F_2P + F_1P = 2a$  where  $a > c$ .

When the point  $P$  describing the ellipse is at  $A_1$ , the sum  $F_2A_1 + F_1A_1 = 2a$ . When  $P$  is at  $A_2$ , the sum  $F_2A_2 + F_1A_2 = 2a$ .

Hence  $F_1A_1 = F_2A_2$ .

Therefore

$$\begin{aligned} F_2P + F_1P &= F_2A_1 + F_1A_1 = F_2A_1 + F_2A_2 = A_2A_1 \\ &= \text{the major axis} = 2a. \end{aligned}$$

That is to say, the **semi-major axis** is  $CA_1 = CA_2 = a$ .

If the minor axis  $B_2B_1$  is denoted by  $2b$ , then the **semi-minor axis** is  $CB_1 = CB_2 = b$ .

Since  $CF_1 = c$ , we have

$$F_2B_1 + F_1B_1 = 2F_1B_1 = 2a,$$

$$\text{or } F_1B_1 = a,$$

$$\text{and } \overline{CF_1} = \overline{F_1B_1} - \overline{CB_1}$$

or

$$(9) \quad c^2 = a^2 - b^2$$

an important relation,  
which shows incidentally  
that  $a > b$ , since  $b \neq 0$ .

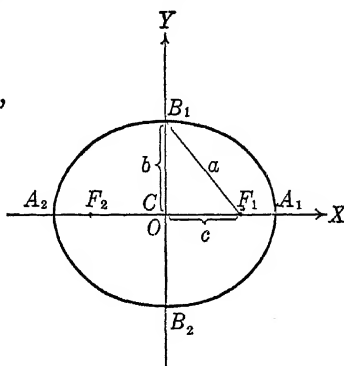


FIG. 72

The ratio  $\frac{c}{a}$  is called the **eccentricity** of the ellipse.

Eccentricity can be used as a measure of the deviation of an ellipse from a circular form. Since  $c < a$ , the ratio  $\frac{c}{a}$  is less than 1.

Hence we have

$$(10) \text{ eccentricity } e = \frac{c}{a},$$

$$(11) \quad e < 1.$$

$$\text{From (9) } c = \sqrt{a^2 - b^2},$$

$$(12) \therefore e = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

A circle is the limiting case of an ellipse whose foci approach each other while the major axis remains constant. Hence  $b$  approaches  $a$ , and  $e$  approaches 0. A circle is often called an **ellipse of eccentricity 0**.

**55. Geometric construction.** If the vertices and foci of an ellipse are given, points on the ellipse are easily determined. Take any arbitrary point  $P$  between  $F_1$  and  $F_2$ . With  $F_1$  as a center and  $PA_1$  as a radius describe arcs above and below the given line. With  $F_2$  as a center and  $PA_2$  as a radius describe arcs above and below the given line intersecting the first set

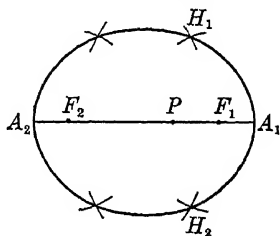


FIG. 73

of arcs in points  $H_1$  and  $H_2$ . These points are on the required ellipse since

$$H_1F_1 + H_1F_2 = PA_1 + PA_2 = 2a.$$

Moreover, four points can be determined for each position of  $P$  by merely reversing the rôles of  $F_1$  and  $F_2$ .

### Exercises

1. The major axis of an ellipse is 10 in. and the minor axis is 6 in. Find the distance between the foci.
2. Construct 12 points on the ellipse for which  $c = 2$  and  $a = 6$ .
3. Construct the ellipse whose semi-axes are 2 cm. and 3 cm.
4. The semi-major axis of an ellipse is 6 cm. and the eccentricity is  $\frac{1}{2}$ . Find the minor axis and the distance between the foci.
5. The eccentricity of an ellipse is  $\frac{3}{5}$  and the distance between the foci is 12 cm. Find the semi-axes.
6. What is the eccentricity of an ellipse whose major axis is twice its minor axis?
7. The foci of an ellipse are at  $(\pm 3, 0)$  and one vertex is at  $(5, 0)$ . Find the eccentricity and semi-minor axis.

**56. The equation of the ellipse.\* Center at origin.**

If we choose the major axis of the ellipse to lie along the  $x$ -axis with the center at the origin, the equation of the ellipse can be derived as follows:

Let  $P(x, y)$  be any point on the ellipse and let the foci be  $F_2(-c, 0)$ ,  $F_1(c, 0)$ .

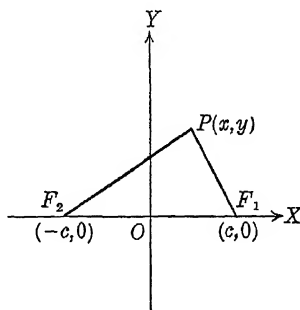


FIG. 74

\* See Ex. 3, p. 37.

From the definition of an ellipse

$$(13) \quad PF_1 + PF_2 = 2a.$$

Hence

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a.$$

Transposing the second radical and squaring, we have

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2,$$

$$(14) \quad \text{or} \quad a\sqrt{(x+c)^2 + y^2} = a^2 + cx$$

Squaring again, we have

$$a^2(x+c)^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2,$$

$$\text{or} \quad (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

$$\text{But} \quad a^2 - c^2 = b^2.$$

$$\therefore \quad b^2x^2 + a^2y^2 = a^2b^2,$$

$$(15) \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We have shown that if the coordinates of a point satisfy (13) they satisfy (15). We must now show that if the coordinates of a point satisfy (15) they satisfy (13) and hence the point is on the ellipse. Assuming the coordinates of a point satisfy (15), we have, by retracing our steps and remembering that when we extract the square root there are two signs, the four equations,

$$\pm \sqrt{(x-c)^2 + y^2} \pm \sqrt{(x+c)^2 + y^2} = 2a.$$

These four equations can be denoted as follows:

$$\begin{array}{ll} (a) & + \quad + \\ (b) & - \quad + \\ (c) & + \quad - \\ (d) & - \quad - \end{array}$$

We wish to show that (a) is the only one of the four equations which is true.

Equations (b) and (c) state that the difference of the distances  $PF_1$  and  $PF_2$  is equal to  $2a$  and hence greater than the third side of the triangle  $PF_1F_2$ , namely  $2c$ . This is absurd, for the difference of two sides of a triangle is always less than the third side.

Equation (d) is false, for the left-hand member is always negative and hence can never equal the positive number  $2a$ . Hence if the coordinates of a point satisfy (15) they satisfy (a) which is the same as (13).

Hence the equation of the required locus is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

**57. Focal radii.** The segments  $PF_1$  and  $PF_2$ , where  $P$  is any point on the ellipse, are called the **focal radii** of the point  $P$ .

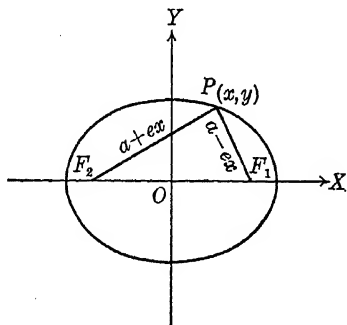


FIG. 75

From equation (14) we have,

$$\sqrt{(x+c)^2 + y^2} = a + \frac{c}{a}x.$$

But

$$e = \frac{c}{a}.$$

$$(16) \quad \therefore PF_2 = \sqrt{(x+c)^2 + y^2} = a + ex.$$

If, in deriving the standard equation we had transposed the first radical and squared, we would have

$$a\sqrt{(x-c)^2 + y^2} = a^2 - cx,$$

$$(17) \text{ Hence } PF_1 = \sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x = a - ex.$$

### Exercises

1. Find the equation of the ellipse, whose semi-axes are 3 in. and 2 in.
2. What is the eccentricity of the ellipse of Ex. 1?
3. Find the coordinates of the foci, the eccentricity, and sketch the ellipse whose equation is  $16x^2 + 25y^2 = 400$ .
4. Same as Ex. 3, for the ellipse whose equation is

$$4x^2 + 25y^2 = 100.$$

5. If the foci of an ellipse are at  $(0, -c)$ ,  $(0, c)$ , the semi-major axis is  $a$  and the semi-minor axis is  $b$ , prove that the equation of the ellipse is  $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ , where  $b^2 + c^2 = a^2$ .
6. Sketch the following ellipses, and on the figure mark the coordinates of the foci and vertices:

$$(a) \quad 4x^2 + 9y^2 = 36. \qquad (b) \quad 9x^2 + 4y^2 = 36.$$

$$(c) \quad 25x^2 + 4y^2 = 100. \qquad (d) \quad 4x^2 + 25y^2 = 100.$$

**58. Directrices. Latus rectum.** From relations (16) and (17) we have

$$r_1 = a - ex,$$

$$r_2 = a + ex.$$

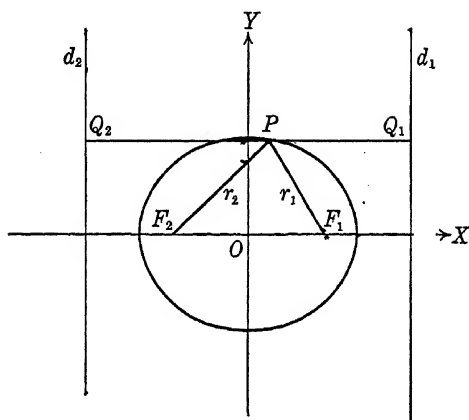


FIG. 76

If we write the first of these in the form  $r_1 = e \left( \frac{a}{e} - x \right)$ ,

and denote by  $d_1$  the line whose equation is  $x = \frac{a}{e}$ , then

$\frac{a}{e} - x$  is the distance  $PQ_1$  from  $P$  to the line  $d_1$  and

$$r_1 = F_1P = e \cdot PQ_1.$$

Similarly,  $r_2 = F_2P = e \left( \frac{a}{e} + x \right)$ , which shows that if

$d_2$  denotes the line whose equation is  $x = -\frac{a}{e}$ , the

distance  $r_2 = F_2P = e \cdot Q_2P$ , where  $Q_2P$  is the distance from  $d_2$  to  $P$ .

The lines  $d_1$  and  $d_2$  are called the **directrices** of the ellipse; each **directrix** being associated with one of the foci,  $d_1$  with  $F_1$  and  $d_2$  with  $F_2$ . In fact, the symmetry of the curve assures the existence of two directrices, after the existence of one has been established.

What we have proved can be stated as follows:

If  $P$  is a variable point on the ellipse, the focal radius to either focus is equal to the eccentricity times the distance of  $P$  from the associated directrix, *i.e.*,

$$F_1P = e \cdot PQ_1; F_2P = e \cdot PQ_2.$$

This property of an ellipse is often used to define the ellipse.

The chord drawn through a focus perpendicular to the major axis is called the **latus rectum** and its length is also called by the same name.

If the equation of the ellipse is  $b^2x^2 + a^2y^2 = a^2b^2$ , the equation of the line through the focus  $(c, 0)$  perpendicular to the major axis is  $x = c$ . It meets the curve in points whose ordinates are given by

$$b^2c^2 + a^2y^2 = a^2b^2.$$

$$y^2 = \frac{a^2b^2 - b^2c^2}{a^2} = \frac{b^2(a^2 - c^2)}{a^2}$$

But from (9),  $a^2 - c^2 = b^2$ .

$$\text{Hence } y^2 = \frac{b^4}{a^2} \text{ or } y = \pm \frac{b^2}{a}.$$

(18)  $\therefore$  The length of the latus rectum is  $2b^2$

*Example.* Given the ellipse whose equation is

$$4x^2 + 9y^2 = 36.$$

Find:

- The semi-axes.
- The coordinates of the foci and the vertices.
- The eccentricity.
- The equations of the directrices.
- The length of the latus rectum and the coordinates of its end points.
- A sketch of the curve.

*Solution:* Writing the equation in standard form we have

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

$$a) \ a^2 = 9, \ b^2 = 4. \quad \therefore \ a = 3, \ b = 2.$$

Semi-major axis = 3, semi-minor axis = 2.

$$b) \ c^2 = a^2 - b^2 = 9 - 4 = 5 \quad = \sqrt{5}.$$

The coordinates of the foci are,  $(-\sqrt{5}, 0), (\sqrt{5}, 0)$ .

The coordinates of the vertices are,  $(-3, 0), (3, 0)$ .

$$c) \ e = \frac{c}{a} = \frac{\sqrt{5}}{3}.$$

*f.* d) The distance from the center to the directrix is

$$\frac{a}{e} = \frac{9}{\sqrt{5}} = \frac{9}{5}\sqrt{5}.$$

$$e = \frac{c}{a}.$$

$$\frac{9}{5} \times \frac{5}{\sqrt{5}} = \frac{9}{\sqrt{5}}$$

$\therefore$  The equations of the directrices are

$$x = -\frac{9}{5}\sqrt{5}, \quad x = \frac{9}{5}\sqrt{5}.$$

e) The length of the latus rectum  $= \frac{2b^2}{a} = \frac{8}{3}$ .

The coordinates of the end points of the latus rectum are  $(-\sqrt{5}, \frac{4}{3})$ ,  $(-\sqrt{5}, -\frac{4}{3})$ ,  $(\sqrt{5}, \frac{4}{3})$ ,  $(\sqrt{5}, -\frac{4}{3})$ .

f)

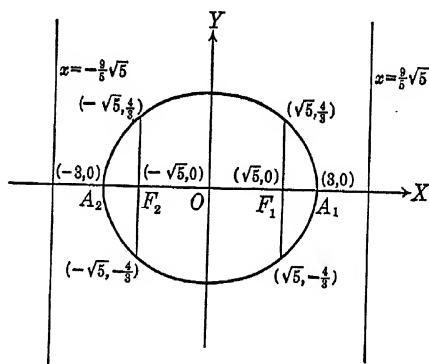


FIG. 77

### Exercises

For the following ellipses find:

- |                      |  |
|----------------------|--|
| a) The semi-axes.    | b) The coordinates of the foci and the vertices. |
| c) The eccentricity. | d) The equations of the directrices.             |

e) The length of the latus rectum and the coordinates of the end points.

f) Sketch of curve.

1.  $x^2 + 4y^2 = 16$ .

2.  $9x^2 + 25y^2 = 225$ .

3.  $4x^2 + y^2 = 36$ .

4.  $9x^2 + 16y^2 = 144$ .

5.  $4x^2 + 3y^2 = 11$ .

6.  $12x^2 + 5y^2 = 23$ .

7.  $4x^2 + y^2 = 16$ .

8.  $25x^2 + 9y^2 = 225$ .

9.  $x^2 + 4y^2 = 36$ .

10.  $16x^2 + 9y^2 = 144$ .

11.  $3x^2 + 4y^2 = 9$ .

12.  $100x^2 + 225y^2 = 324$ .

13. Find the focal radii for the point  $(3, 2)$  for the ellipse whose equation is  $x^2 + 4y^2 = 25$ .

14. Find the equation of the ellipse whose vertices are the points  $(-3, 0)$ ,  $(3, 0)$  and which passes through the point  $(2, 1)$ .

15. Find the equation of the ellipse whose vertices are the points  $(0, -4)$ ,  $(0, 4)$  and which passes through the point  $(2, 3)$ .

16. Find the equation of the ellipse whose foci are the points  $(-3, 0)$ ,  $(3, 0)$  and whose minor axis is 8.

17. Find the equation of the ellipse whose foci are at the points  $(-3, 0)$ ,  $(3, 0)$  and whose major axis is twice the minor axis.

18. Find the equation of the ellipse, foci at  $(-3, 0)$ ,  $(3, 0)$ , and eccentricity  $\frac{2}{3}$ .

19. Find the equation of the ellipse, vertices at  $(0, -5)$ ,  $(0, 5)$ , and eccentricity  $\frac{1}{2}$ .

20. Find the equation of the ellipse, foci at  $(-2, 0)$ ,  $(2, 0)$ , latus rectum 6.

21. Find the equation of the ellipse whose eccentricity is  $\frac{1}{2}$ , and the equations of whose directrices are  $x = 4$ ,  $x = -4$ .
22. Find the equation of the ellipse whose eccentricity is  $\frac{2}{3}$  and the equation of whose directrices are  $y = 5$ ,  $y = -5$ .
23. Find the coordinates of the points of intersection of  $x^2 + y^2 = 25$ ,  $2x^2 + y^2 = 41$ . Sketch the curves.
24. Find the coordinates of the points of intersection of  $x^2 + y^2 = 16$ ,  $4x^2 + y^2 = 16$ . Sketch the curves.
25. Find the equation of the ellipse whose center is at  $(0, 0)$  and which pass through the points  $(-2, -2)$ ,  $(-3, 1)$ .
26. If  $b$  approaches  $a$  as a limit, what curve does the ellipse approach as a limit? What is the limit of  $e$ ? of  $c$ ? of  $\frac{2b^2}{a}$ ?  $\frac{a}{e}$ ?

**59. A hyperbola** is the locus of a point in a plane such that the difference of its distances in either order from two fixed points in the plane, called foci, is a positive constant less than the distance between the foci.

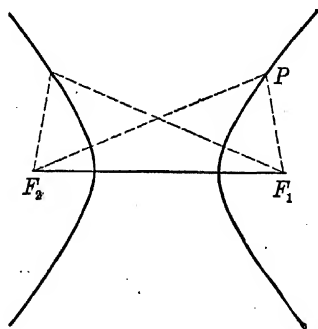


FIG. 78

If the foci are  $F_1$ ,  $F_2$ , the moving point  $P$ , and if the constant be denoted by  $2a$ , the definition requires that

$$PF_2 - PF_1 = 2a,$$

or

$$PF_1 - PF_2 = 2a.$$

If we denote the length of the segment  $F_1F_2$  by  $2c$  then  $2c > 2a$  by the definition of a hyperbola. This agrees with the plane geometry theorem, "The length of any side of a plane triangle is greater than the difference of the lengths of the other two sides."

The following simple mechanical device makes it possible for us to draw a hyperbola. Place pegs or thumb tacks at the foci  $F_1$  and  $F_2$ . Pass a string over  $F_2$  and

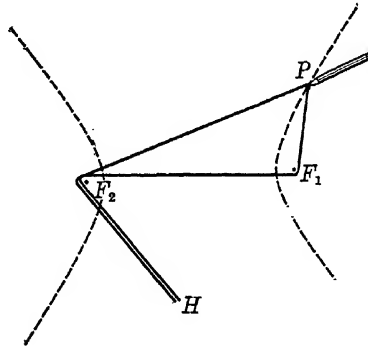


FIG. 79

around  $F_1$  and let the ends be held together at  $H$ . If a pencil be tied to the string at  $P$  and if the string be kept taut by the pencil, then if the ends be pulled in or let out together,  $PF_2 - PF_1$  is constant and hence  $P$  describes a hyperbola.

## 60. Geometric construction of a hyperbola. Let

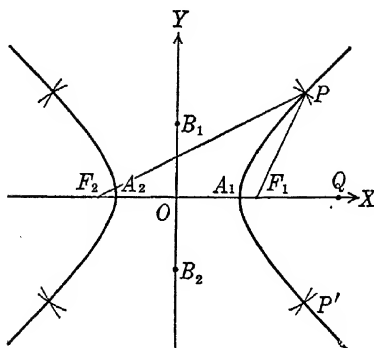


FIG. 80

$F_1, F_2$  be the foci and draw the line through them. Let  $O$  be the mid-point of  $F_1F_2$  and let  $A_1, A_2$  be the two points on the line through  $F_1$  and  $F_2$  on either side of  $O$  at a distance  $a$  from  $O$ . The point  $A_1$  is then a point of the hyperbola, for  $A_1F_1 = c - a$  and  $A_1F_2 = c + a$ , so that  $A_1F_2 - A_1F_1 = 2a$ . Similarly,  $A_2$  is a point on the locus. Now let  $Q$  be any point to the right of  $F_1$  on the line through  $F_1$  and  $F_2$ . Then  $A_2Q - A_1Q = A_2A_1 = 2a$ . If, then, with  $F_1$  as center and  $A_1Q$  as radius a circle be described, and with  $F_2$  as center and  $A_2Q$  as a radius another circle be described, the two circles will intersect in  $P$  and  $P'$ , which are points on the hyperbola. With the same radii and interchanging the centers  $F_1$  and  $F_2$  two more points can be found. From every position of  $Q$  we thus get four points of our curve; by choosing various positions for  $Q$  any desired number of points of the hyperbola can be located.

This construction shows that the hyperbola consists

of two branches, each symmetric with respect to the line  $F_2F_1$ , and each the reflection of the other in the perpendicular bisector,  $B_2B_1$  of  $F_2F_1$ . The hyperbola as a whole is then symmetric with respect to both of these lines, which are appropriately called **axes** of the curve. The line through  $F_2$  and  $F_1$  cuts the curve in the two **vertices**  $A_1$  and  $A_2$ . The undirected segment  $A_2A_1$  is called the **transverse axis** of the curve. It is of length  $2a$ . The other line of symmetry  $B_2B_1$  has no point in common with the curve. However, the term **conjugate axis** will be applied to the undirected segment  $B_2B_1$ . The points  $B_1$  and  $B_2$  are so located that  $\overline{OF_1}^2 = \overline{OA_1}^2 + \overline{OB_1}^2$  and  $B_2O = OB_1$ . We shall denote the length of  $B_2O = OB_1$ , by  $b$ .

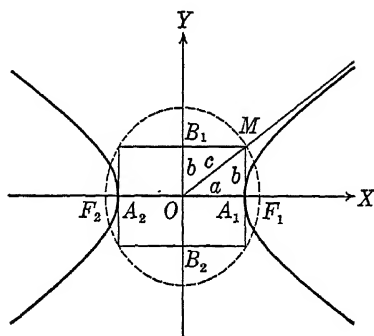


FIG. 81

(19) Therefore  $c^2 = a^2 + b^2$ .

From (19) it follows that  $b$  may be less than, equal to, or greater than  $a$ .

The ratio  $\frac{c}{a}$  is called the **eccentricity** of the hyperbola.

Since  $c > a$ , the eccentricity is always greater than **one**.

Hence,

$$(20) \quad \text{eccentricity } e = \frac{c}{a},$$

$$(21) \quad e > 1.$$

### Exercises

1. The transverse axis of a hyperbola is 10 in. and the conjugate axis is 12 in. Find the distance between the foci.
2. Construct 12 points on the hyperbola for which  $c = 6$ ,  $a = 4$ .
3. The semi-transverse axis of a hyperbola is 6 cm. and the eccentricity is  $\frac{3}{2}$ . Find the conjugate axis and the distance between the foci.
4. The eccentricity of a hyperbola is  $\frac{3}{2}$  and the distance between the foci is 10 cm. Find the semi-axes.
5. What is the eccentricity of a hyperbola whose transverse axis is equal to its conjugate axis?
6. The foci of a hyperbola are at  $(\pm 3, 0)$  and one vertex is at  $(2, 0)$ . Find the eccentricity and the semi-conjugate axis.

**61. The equation of the hyperbola. Center at origin.**  
The center is at  $(0, 0)$  and let the foci be  $F_1(c, 0)$ ,  $F_2(-c, 0)$ .

Let  $P(x, y)$  be any point on the hyperbola. From the definition of a hyperbola

$$(22) \quad PF_1 - PF_2 = 2a,$$

or

$$(23) \quad PF_2 - PF_1 = 2a.$$

Algebraically, we have

$$\begin{aligned} \sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} &= 2a \quad \text{or} \\ \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} &= 2a. \end{aligned}$$

If we clear these equations of radicals as we did in the case of the ellipse, § 56, we have, recalling that

$$(24) \quad \frac{b^2}{a^2} - \frac{y^2}{c^2} = 1,$$

If the coordinates of a point  $P(x, y)$  satisfy (24) we must show they satisfy either (22) or (23). Assuming the coordinates of a point satisfy (24) we have, by retracing our steps and remembering that when we extract the square root that there are two signs, the four equations

$$\pm \sqrt{(x+c)^2 + y^2} \pm \sqrt{(x-c)^2 + y^2} = 2a.$$

These four equations can be denoted as follows:

$$\begin{array}{ll} a) & + \quad - \\ b) & - \quad + \\ c) & + \quad + \\ d) & - \quad - \end{array}$$

Equations  $a)$  and  $b)$  are precisely equations (22) and (23). Equation  $c)$  is false, for it says the sum of the two sides  $PF_1$  and  $PF_2$  of triangle  $PF_1F_2$  is equal to  $2a$ , which by hypothesis is less than  $2c$ , the third side of the triangle, or in other words, we have the sum of two sides of a triangle less than the third side, which is impossible. Equation  $d)$  is false, for it says the sum of two

negative numbers is equal to a positive number. Hence, if the coordinates of a point satisfy (24) they satisfy (a) or (b) which is the same as (22) or (23). Therefore the equation of the required locus is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

**62. Focal Radii.** The segments  $PF_1$  and  $PF_2$ , where  $P$  is any point on the ellipse, are called the focal radii of the point  $P$ .

Proceeding as in the case of the ellipse, § 57, we have, when  $P$  is a point on the right-hand branch,

$$PF_2 = ex + a,$$

$$PF_1 = ex - a.$$

When  $P$  is a point on the left-hand branch,

$$PF_2 = a - ex,$$

$$PF_1 = -a - ex.$$

### Exercises

1. Find the equation of the hyperbola whose semi-axes are 3 in. and 4 in.
2. What is the eccentricity of the hyperbola in Ex. 1?
3. Find the coordinates of the foci, the eccentricity, and sketch the curve, of the hyperbola whose equation is  $16x^2 - 25y^2 = 400$ .
4. Same as Ex. 3, for the hyperbola whose equation is  $4x^2 - 25y^2 = 100$ .

5. If the foci of a hyperbola are at  $(0, -c)$ ,  $(0, c)$ , the semi-major axis is  $a$ , and the semi-conjugate axis is  $b$ , prove that the equation of the hyperbola is  $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ , where  $c^2 = a^2 + b^2$ .

6. Sketch the hyperbolas with the following equations:

a)  $4x^2 - 9y^2 = 36$ .

b)  $9x^2 - 4y^2 = 36$ .

c)  $25y^2 - 4x^2 = 100$ .

d)  $4y^2 - x^2 = 16$ .

**63. Directrices. Latus rectum.** Proceeding as we did in the case of the ellipse (§ 57), the focal radii  $r_1$  and  $r_2$  can be written

$$r_1 = ex - a = e \cdot \left( x - \frac{a}{e} \right),$$

$$(25) \quad r_2 = ex + a = e \cdot \left( x + \frac{a}{e} \right)$$

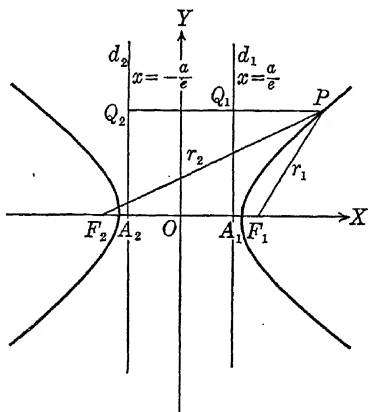


FIG. 82

But  $x - \frac{a}{e}$  is the distance  $Q_1P$  of  $P$  from the line  $d_1$

whose equation is  $x - \frac{a}{e} = 0$ ; and  $x + \frac{a}{e}$  is the distance

$Q_2P$  of  $P$  from the line  $d_2$  whose equation is  $x + \frac{a}{e} = 0$ .

Equations (25) then state

$$\begin{aligned} r_1 &= F_1P = e \cdot Q_1P, \\ (26) \quad r_2 &= F_2P = e \cdot Q_2P. \end{aligned}$$

The lines  $d_1$  and  $d_2$  are called the **directrices** of the hyperbola, each **directrix** being associated with one of the foci;  $d_1$  with  $F_1$ , and  $d_2$  with  $F_2$ . From the derivation equations (25) hold only when  $r_2 > r_1$ , that is to say, when  $x > 0$ . But the symmetry of the curve with respect to the  $y$ -axis insures that the relations hold on the other branch of the hyperbola, namely, when  $x < 0$ . What we have proved can be stated as follows:

If  $P$  is a variable point on a hyperbola, the focal radius to either focus is equal to the eccentricity times the distance of  $P$  from the associated directrix; i.e.,  $F_1P = e \cdot Q_1P$  and  $F_2P = e \cdot Q_2P$ .

The chord of the hyperbola drawn through a focus perpendicular to the transverse axis or its length is called the **latus rectum**. It is left to the student to show that its length is  $\frac{2b^2}{a}$ .

**64. The asymptotes.** We are already familiar in a general way with the shape of a hyperbola. We know it consists of two branches, and that it is symmetric

with respect to two perpendicular lines. There is one feature of the shape of this curve which requires more careful consideration. Let the equation of the curve

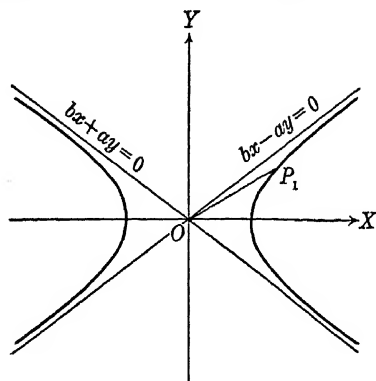


FIG. 83

be  $b^2x^2 - a^2y^2 = a^2b^2$  and let us consider the line  $OP_1$ , joining the center  $O$  to a point  $P_1(x_1, y_1)$  on the curve. The slope  $m_1$  of  $OP_1$  is  $y_1/x_1$ . Since  $P_1$  is on the curve,

$$(27) \quad b^2x_1^2 - a^2y_1^2 = a^2b^2,$$

or

$$y_1^2 = \frac{b^2}{a^2} (x_1^2 - a^2),$$

from which

$$\frac{y_1^2}{x_1^2} = \frac{b^2}{a^2} \left( 1 - \frac{a^2}{x_1^2} \right),$$

so that

$$m_1 = \frac{y_1}{x_1} = \pm \frac{b}{a} \sqrt{1 - \frac{a^2}{x_1^2}}.$$

To fix ideas, let us now suppose that  $P_1$  is in the first quadrant, and that  $P_1$  recedes indefinitely on that branch

of the curve. The slope  $m_1$  of  $OP_1$  is positive, and as  $x_1$  increases indefinitely the value of  $m_1$  approaches more and more nearly the value  $\frac{b}{a}$ , since  $\frac{a^2}{x_1^2}$  becomes more and more nearly equal to zero. The line  $OP_1$  then approaches a limiting position of the line whose slope is  $b/a$  and whose equation is

$$(28) \quad bx - ay = 0.$$

Similarly, if  $P_1$  is on the part of the curve in the second quadrant and recedes indefinitely in this quadrant, the line  $OP_1$  approaches a limiting position whose slope is  $-b/a$  and whose equation is

$$(29) \quad bx + ay = 0.$$

The two lines (28) and (29) have a special relation to the hyperbola. Let  $d_1$  and  $d_2$  denote the distances of  $P_1$  from these lines. Then

$$d_1 = \frac{bx_1 - ay_1}{\sqrt{a^2 + b^2}}, \quad d_2 = \frac{bx_1 + ay_1}{\sqrt{a^2 + b^2}}.$$

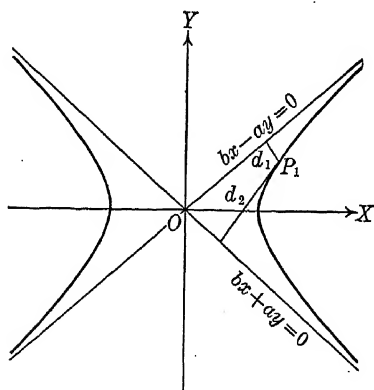


FIG. 84

The product of these distances is

$$(30) \quad d_1 d_2 = \frac{bx_1 - ay_1}{\sqrt{a^2 + b^2}} \cdot \frac{bx_1 + ay_1}{\sqrt{a^2 + b^2}} = \frac{b^2 x_1^2 - a^2 y_1^2}{a^2 + b^2} = \frac{a^2 b^2}{a^2 + b^2}$$

from (27). As  $P_1$  moves on the curve, the product  $d_1 d_2$  remains constant. Now let  $P_1$  lie in the first quadrant and move out along the curve. It now surely recedes indefinitely from the line  $bx + ay = 0$  which lies in the second and fourth quadrants. When  $P_1$  recedes indefinitely from this line it approaches more and more closely the line  $bx - ay = 0$ . More precisely, if the constant  $\frac{a^2 b^2}{(a^2 + b^2)}$  be denoted by  $k^2$ , we have  $d_1 = \frac{k^2}{d_2}$ .

Hence, by letting  $P_1$  recede far enough, that is, by making  $d_1$  sufficiently large,  $d_2$  can be made as small as we please. We get an exactly similar situation if  $P_1$  moves out on the hyperbola in any of the other quadrants.

If a fixed straight line is so related to an infinite branch of a curve that as a point on the curve recedes indefinitely along the infinite branch, the distance of  $P$  from the line approaches zero, the line is called an *asymptote\** of the curve.

**A hyperbola has two asymptotes.**

We have also proved that the product of the distances of any point on a hyperbola from its asymptotes is constant.

It should be noted that the asymptotes of  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  have as equations the factors of  $\frac{x^2}{a^2} - \frac{y^2}{b^2}$ , placed equal to 0.

\*See § 23.

**65. Construction of the asymptotes.** Through the extremities of the major and minor axes, draw lines parallel to the other axis, thus forming a rectangle. The diagonals of the rectangle, produced, are the asymptotes, since they pass through the origin and have slopes  $\pm \frac{b}{a}$ .

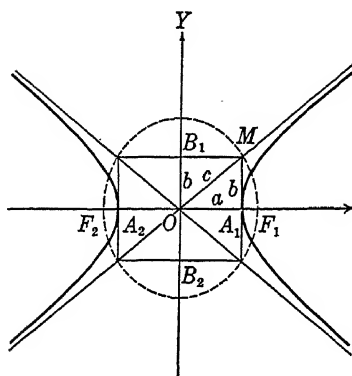


FIG. 85

Since (see Fig. 85)  $OA_1 = a$ ,  $A_1M = b$ , it follows that  $OM = c$ , since from § 60,  $a^2 + b^2 = c^2$ .

$\therefore$  We can locate the foci by drawing a circle, center  $O$ , radius  $c$ , and finding the points where it meets  $A_1A_2$  prolonged.

*Example:* Given the hyperbola whose equation is  $4x^2 - 9y^2 = 36$ . Find:

- The semi-axes.
- The coordinates of the foci and the vertices.
- The eccentricity.

- d) The equations of the directrices.  
 e) The equations of the asymptotes.  
 f) The length of the latus rectum and the coordinates of the end points.  
 g) Sketch of curve.

Solution: Writing the equation in standard form we have

$$\frac{x^2}{9} - \frac{y^2}{4} = 1.$$

a)  $a^2 = 9, b^2 = 4. \therefore a = 3, b = 2.$

b)  $c^2 = a^2 + b^2 = 13, c = \sqrt{13}.$

Coordinates of foci are  $(-\sqrt{13}, 0), (\sqrt{13}, 0).$

Coordinates of vertices are  $(-3, 0), (3, 0).$

c)  $e = \frac{c}{a} = \frac{\sqrt{13}}{3}.$

d) Distance from center to directrix is

$$\frac{a}{e} = \frac{9}{\sqrt{13}} = \frac{9}{13}\sqrt{13}.$$

$\therefore$  Equations of directrices are

$$x = \frac{9\sqrt{13}}{13}, \quad x = -\frac{9\sqrt{13}}{13}.$$

e) The equations of the asymptotes are the factors of  $4x^2 - 9y^2$  placed equal to 0.

$$\therefore 2x - 3y = 0, \quad 2x + 3y = 0.$$

f) The latus rectum  $= \frac{2b^2}{a} =$

The coordinates of the end points are  $(-\sqrt{13}, \frac{4}{3})$ ,  $(-\sqrt{13}, -\frac{4}{3})$ ,  $(\sqrt{13}, \frac{4}{3})$ ,  $(\sqrt{13}, -\frac{4}{3})$ .

- g) Lay off  $OA_1 = OA_2 = 3$ ,  $OB_1 = OB_2 = 2$  and construct the rectangle and asymptotes as described in § 65. The curve can then be readily sketched by noting that  $A_1$  and  $A_2$  are the vertices.

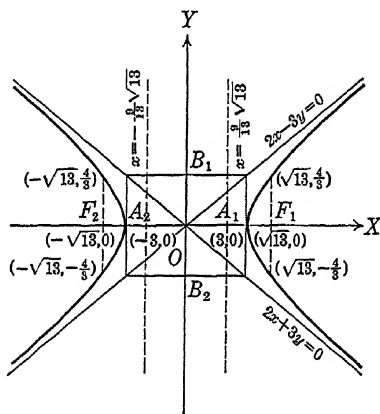


FIG. 86

### Exercises

For the following hyperbolas find:

- The semi-axes.
- The coordinates of the foci and the vertices.
- The eccentricity.
- The equations of the directrices.
- The equations of the asymptotes.

- f) The length of the latus rectum and the coordinates of the end points.
- g) Sketch of the curve.
1.  $9x^2 - 16y^2 = 144$ .
  2.  $16x^2 - 9y^2 = 144$ .
  3.  $4x^2 - 9y^2 = 36$ .
  4.  $x^2 - y^2 = 4$ .
  5.  $25x^2 - 144y^2 = 3600$ .
  6.  $4x^2 - 9y^2 = 25$ .
  7.  $16y^2 - x^2 = 16$ .
  8.  $16y^2 - 9x^2 = 144$ .
  9.  $y^2 - 4x^2 = 36$ .
  10.  $9y^2 - 25x^2 = 225$ .
  11.  $y^2 - x^2 = 16$ .
  12.  $4y^2 - 9x^2 = 25$ .
13. Find the equation of the hyperbola, center at origin, transverse axis = 8 along the  $x$ -axis, conjugate axis = 6.
14. Find the equation of the hyperbola, foci at  $(\pm 5, 0)$ . Conjugate axis = 8.
15. Find the equation of the hyperbola, center at origin, one focus at  $(5, 0)$ , eccentricity =  $\frac{5}{3}$ .
16. Find the equation of the hyperbola whose vertices are  $(\pm 4, 0)$  and which passes through  $(5, 4)$ .
17. Find the equation of the hyperbola whose foci are  $(\pm 3, 0)$  and whose conjugate axis is 4.
18. Find the equation of the hyperbola whose foci are  $(\pm 3, 0)$  and whose transverse axis is twice its conjugate axis.
19. Find the equation of the hyperbola, foci at  $(\pm 3, 0)$ , eccentricity  $\frac{3}{2}$ .
20. Find the equation of the hyperbola, vertices at  $(0, \pm 5)$ , eccentricity  $\frac{4}{3}$ .
21. Find the equation of the hyperbola, foci at  $(\pm 2, 0)$ , latus rectum = 6.

22. Find the equation of the hyperbola, center at the origin, whose eccentricity is  $\frac{5}{3}$  and  $x = 2$  the equation of a directrix.
23. Find the equation of the hyperbola, center at the origin, whose eccentricity is  $\frac{5}{3}$  and  $y = 3$  the equation of a directrix.
24. Find the equation of the hyperbola whose asymptotes are  $2x - y = 0$ ,  $2x + y = 0$  and which passes through  $(3, 4)$ .
25. Find the equation of the hyperbola whose asymptotes are  $3x - y = 0$ ,  $3x + y = 0$  and one of whose vertices is  $(4, 0)$ .
26. Find the equation of the hyperbola whose asymptotes are  $x - 3y = 0$ ,  $x + 3y = 0$  and one of whose foci is  $(4, 0)$ .
27. If the ratio of the conjugate axis to the transverse axis is  $k$ , prove  $e^2 = 1 + k^2$ .

**66. Conjugate hyperbolas.** A given hyperbola always determines another, its so-called **conjugate hyperbola**, the transverse axis of which is the conjugate axis of

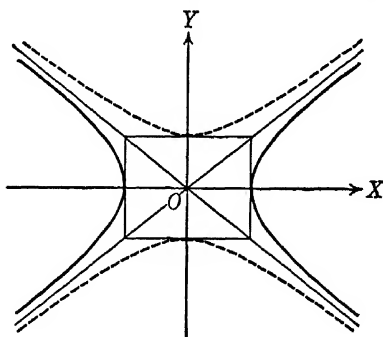


FIG. 87

the given one, while its conjugate axis is the transverse axis of the given one. Two conjugate hyperbolas have then

the same asymptotes. If the equation of a hyperbola is given, the equation of its conjugate is obtained by changing the sign of the constant. Thus, if  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is the equation of the given hyperbola,  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$  is the equation of the conjugate hyperbola.

The equations of the asymptotes for both hyperbolas are

$$\frac{x}{a} - \frac{y}{b} = 0, \quad \frac{x}{a} + \frac{y}{b} = 0.$$

**67. Equilateral or rectangular hyperbola.** A hyperbola in which the transverse and conjugate axes are equal is called **equilateral**. The condition  $b = a$  therefore gives as the equation of an equilateral hyperbola, center at origin and transverse axis along the  $x$ -axis,

$$(31) \quad x^2 - y^2 = a^2.$$

Its conjugate hyperbola has the equation

$$x^2 - y^2 = -a^2,$$

while the equations of their common asymptotes are given by

$$x^2 - y^2 = 0.$$

The latter are at right angles to each other. Conversely, the slopes  $m_1, m_2$  of the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ are } m_1 = \frac{b}{a}, \quad m_2 = -\frac{b}{a}. \quad \text{If these are per-}$$

pendicular

$$m_1 m_2 = -\frac{b^2}{a^2} = -1.$$

This gives  $b^2 = a^2$ . A hyperbola whose asymptotes are perpendicular is called **rectangular**. We have just seen that a rectangular hyperbola is equilateral, and conversely.

If the asymptotes of a rectangular hyperbola are taken as coordinate axes, the equation of the hyperbola assumes an especially simple form. The distances,  $d_1, d_2$  of a point  $P(x, y)$  from the asymptotes are now  $x, y$ , respectively. Equation (30) gives in this case

$$xy = \frac{a^2 b^2}{a^2 + b^2},$$

or since  $b = a$ ,

$$(32) \quad xy =$$

If the hyperbola lies in the first and third quadrants,  $xy$  is positive; if the hyperbola lies in the second and fourth quadrants,  $xy$  is negative. Hence, the equation of a rectangular hyperbola referred to its asymptotes as coordinate axes is

$$xy = \frac{a^2}{2}$$

if the curve is in the first and third quadrants; and is

$$xy = -\frac{a^2}{2},$$

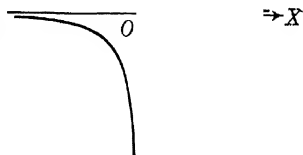


FIG. 88

if it is in the second and fourth quadrants.

## Exercises

Find the equation of the hyperbola conjugate to each of the following:

1.  $9x^2 - 16y^2 + 144 = 0$ .
2.  $y^2 = 4(x^2 - 1)$ .
3.  $xy = 10$ .
4.  $xy = -k$ .
5. Find the equation of the hyperbola conjugate to the hyperbola whose asymptotes are  $2x - y = 0$  and  $2x + y = 0$  and which passes through the point  $(2, 6)$ .
6. Find the equation of the hyperbola whose asymptotes are the coordinate axes and which passes through the point  $(-2, -4)$ . What are the coordinates of its vertices?
7. A hyperbola has its axes lying along the coordinate axes and passes through the points  $(2, 0)$  and  $(4, 4)$ ; find its equation.
8. What is the eccentricity of an equilateral hyperbola?

**68. Alternate definition of a conic.** A conic is either a circle or the locus of a point such that the ratio of its distance from a fixed point in the plane, called the focus, to its distance from a fixed line in the plane not passing through the focus, called the directrix, is equal to a positive constant  $k$  which is not zero.

Take the directrix as the  $y$ -axis and the  $x$ -axis to be the perpendicular to the  $y$ -axis which passes through the focus. Let  $OF = p$ , and let  $P(x, y)$  be any point on the locus. Then

$$\frac{PF}{PD} = k. \quad k > 0.$$

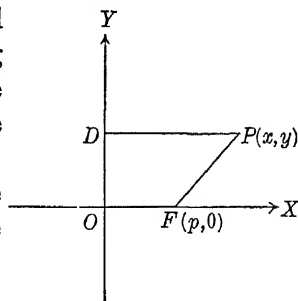


FIG. 89

But  $PF = \sqrt{(x-p)^2 + y^2}$ ,  $PD = x$ .

$$\frac{\sqrt{(x-p)^2 + y^2}}{x} = k.$$

Squaring and simplifying, we have

$$(1-k)^2 x^2 + y^2 - 2px + p^2 = 0,$$

as the equation of the desired locus. From this equation we see that;

1. When  $k = 1$ , the locus is a parabola.
2. When  $k < 1$ , the locus is an ellipse.
3. When  $k > 1$ , the locus is a hyperbola.

In other words, our locus is the general equation of a conic.

We shall now show that the constant  $k$  is the eccentricity. When  $k = 1$ , it is evident from § 50 that the eccentricity is  $k$ . When  $k < 1$ , by completing the square the equation can be written in the form

$$\frac{\left(x - \frac{p}{1-k^2}\right)^2}{\frac{p^2 k^2}{(1-k^2)^2}} + \frac{y^2}{\frac{p^2 k^2}{1-k^2}} = 1.$$

$$\therefore a = \frac{pk}{1-k^2}, \quad b = \frac{pk}{\sqrt{1-k^2}} \text{ and } c = \frac{pk^2}{1-k^2}.$$

$$\text{But the eccentricity} = \frac{c}{a} = \frac{pk^2}{1-k^2} \div \frac{pk}{1-k^2} = k.$$

It is left as an exercise for the student to prove that the eccentricity is  $k$  when  $k > 1$ .

## MISCELLANEOUS EXERCISES

1. Find the equation of the parabola, vertex at origin, axis of curve along the  $y$ -axis, and passing through  $(3, 4)$ .
2. Show that the latus rectum of an ellipse is  $2b\sqrt{1-e^2}$  and  $2a(1-e^2)$ .
3. Prove that the minor axis of an ellipse is a mean proportional between the major axis and the latus rectum.
4. Prove that in an ellipse the major axis is a mean proportional between the distance between the foci and the distance between the directrices.
5. Find the equation of the equilateral hyperbola whose foci are  $(\pm 3, 0)$ .
6. Find the equation of the equilateral hyperbola, center at the origin, cutting the  $x$ -axis, and passing through  $(5, 3)$ .
7. Prove that  $x^2 + 3y^2 = 24$  and  $3x^2 - y^2 = 12$  have the same foci.
8. Prove that all of the hyperbolas of the system  $4x^2 - 9y^2 = k$  have the same asymptotes.
9. Prove that the vertices of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  subtend a right angle at the points  $(0, \pm b)$ , when and only when the hyperbola is rectangular.
10. Prove that in a hyperbola, an asymptote, the line through a focus perpendicular to the asymptote and the directrix corresponding to the focus are concurrent.
11. Prove that a line through the vertex of a parabola and making with the axis of the curve an angle whose tangent is 2, passes through an end of the latus rectum.

## CHAPTER VII

### TANGENTS

69. Consider any curve and a fixed point  $P_1$  on it. Any line through  $P_1$  and some other point  $P_2$  on the curve is called a **secant**. If  $P_2$  moves along the curve toward  $P_1$ , the secant will rotate about  $P_1$ , and as  $P_2$  approaches  $P_1$ , the secant will usually approach a **limiting position**, which is defined to be the **tangent to the curve at  $P_1$** .

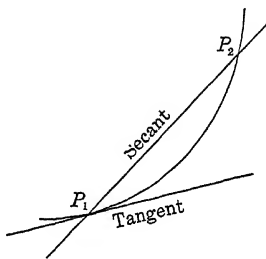


FIG. 90

If  $P_1P_2$  does not approach a limiting position the curve has no tangent at  $P_1$ .

In any particular case we find the slope of the secant through  $P_1$  and a second point  $P_2$ . The limit of this expression as  $P_2$  approaches  $P_1$  along the curve, provided such a limit exists, is called the **slope of the tangent at  $P_1$** . The slope of the tangent to the curve at any point  $P$  is called the **slope of the curve at  $P$** .

The method of finding the equation of the tangent to a curve at a particular point will be illustrated by several examples.

*Example 1.* Find the equation of the tangent to  $y = x^2$  at  $(x_1, y_1)$ .

Solution: We must find the slope of the tangent at  $(x_1, y_1)$ . To this end we shall proceed as follows.

Let a secant through  $P_1$  cut the curve in the point  $P_2$ . If  $P_1Q = h$ ,  $QP_2 = k$ , then the coordinates of point  $P_2$  are  $(x_1 + h, y_1 + k)$ .

In the first column we have the analytic statement corresponding to the like numbered statement in the second column.

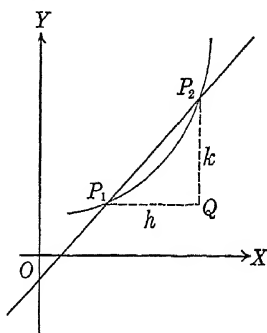


FIG. 91

- |                                  |  |
|----------------------------------|--|
| 1. $y_1 + k = (x_1 + h)^2$ ,     | 1. Since $P_2$ is on the curve its coordinates satisfy the given equation.   |
| $\quad = x_1^2 + 2hx_1 + h^2$ .  |  |
| 2. $y_1 = x_1^2$ .               | 2. Since $P_1$ is on the curve its coordinates satisfy the given equation.   |
| 3. $k = 2hx_1 + h^2$ ,           | 3. Subtracting (2) from (1) and dividing by $h$ , we have the slope $\frac{k}{h}$ of the secant $P_1P_2$ .   |
| $\quad \frac{k}{h} = 2x_1 + h$ . |  |
| 4. $m = 2x_1$ .                  | 4. As $P_2$ approaches $P_1$ along the curve, $h$ and $k$ approach zero. The limit of $\frac{k}{h}$ as $h$ approaches zero,* is the slope of the tangent line at $P_1$ . |

$\therefore$  The equation of the tangent at  $P_1$  is

$$y - y_1 = 2x_1(x - x_1).$$

\* The students should note that when  $h$  approaches zero,  $k$  does also.

If we wish the equation of the tangent at a particular point, say,  $(2, 4)$  we have

$$m = 2x_1 = 4.$$

The equation of the tangent is,  $y - 4 = 4(x - 2)$ ,

or 
$$y - 4x + 4 = 0.$$

*Example 2.* Find the equation of the tangent to  $3x^2 - 2y^2 = 10$  at the point  $(2, 1)$ .

*Solution:* We shall first find the slope of the tangent at any point  $(x_1, y_1)$  as we did in Example 1.

Let  $P_1(x_1, y_1)$  and  $P_2(x_1 + h, y_1 + k)$  be two points on the same branch of the curve.

- |  |  |
|--|--|
| 1. $3(x_1 + h)^2 - 2(y_1 + k)^2 = 10,$<br>$3x_1^2 + 6x_1h + 3h^2 - 2y_1^2 -$<br>$4y_1k - 2k^2 = 10.$   | 1. Since $P_2$ is on the curve its coordinates satisfy the given equation.   |
| 2. $3x_1^2 - 2y_1^2 = 10.$   | 2. Since $P_1$ is on the curve its coordinates satisfy the given equation.   |
| 3. $6x_1h + 3h^2 - 4y_1k - 2k^2 = 0,$<br>$h(6x_1 + 3h) - k(4y_1 + 2k) = 0.$<br>$(6x_1 + 3h) - \frac{k}{h}(4y_1 + 2k) = 0,$<br>$\frac{k}{h} = \frac{6x_1 + 3h}{4y_1 + 2k}.$ | 3. Subtracting (2) from (1), simplifying, dividing by $h$ , solving for $\frac{k}{h}$ , we have the slope of the secant $P_1P_2$ . |

$$4. m = \frac{6x_1}{4y_1} = \frac{3x_1}{2y_1}.$$

4. As  $P_2$  approaches  $P_1$  along the curve,  $h$  and  $k$  approach zero. The limit of  $\frac{k}{h}$  as  $h$  approaches zero is the slope of the tangent line at  $P_1$ .

At the point  $(2, 1)$ ,  $m = 3$ .

Hence, the equation of the tangent is  $y - 1 = 3(x - 2)$ ,

or  $y - 3x + 5 = 0$ .

*Example 3.* Prove that the equation of the tangent to

$$y^2 = 4px \text{ at } (x_1, y_1) \text{ is } y_1y = 2p(x + x_1).$$

*Solution:* Proceeding as in the last two examples we have

$$1. (y_1 + k)^2 = 4p(x_1 + h),$$

$$y_1^2 + 2ky_1 + k^2 = 4px_1 + 4ph.$$

$$2. y_1^2 = 4px_1.$$

$$3. 2ky_1 + k^2 = 4ph,$$

$$k(2y_1 + k) = 4ph,$$

$$\frac{k}{h} = \frac{4p}{2y_1 + k}.$$

$$4. \text{ Hence } m = \frac{4p}{2y_1} = \frac{2p}{y_1}.$$

Therefore the equation of the tangent is

$$y - y_1 = \frac{2p}{y_1}(x - x_1),$$

or  $y_1 y - y_1^2 = 2 p x - 2 p x_1.$

From (2)  $y_1^2 = 4 p x_1$ ; therefore

$$y_1 y - 4 p x_1 = 2 p x - 2 p x_1,$$

or  $y_1 y = 2 p (x + x_1).$

### Exercises

Find the slope of the following curves at the point  $(x_1, y_1)$ :

1.  $y = x^2 + 5.$                       2.  $y = x^2 - 3 x + 1.$

3.  $y = 2 x^2 - x - 1.$                 4.  $y^2 = 2 x - 1.$

5.  $x^2 - 2 y = 7.$                       6.  $2 x^2 + 3 y = 8.$

Find the equations of the tangents to the following curves at the points indicated:

7.  $y = 2 x^2, \quad (1, 2).$               8.  $y = 3 x^2 + 1, \quad (1, 4).$

9.  $y = \frac{1}{x} \quad (2, \frac{1}{2}).$               10.  $y = \frac{3}{x-5} \quad (6, 3).$

11.  $y = x^3 - 4 x, \quad (2, 0).$

12.  $y = x^3 - 3 x^2 + 6 x - 1, \quad (1, 3).$

13. If  $n$  is a positive integer, prove that the slope of the curve  $y = c x^n$  at point  $(x_1, y_1)$  is  $c n x_1^{n-1}.$

**70. Tangents to a conic.** The following results are given without proof.

I. The slope of the parabola  $y^2 = 4 p x$  at the point  $(x_1, y_1)$  is

$$m = \frac{2 p}{y_1}.$$

The equation of the tangent at  $(x_1, y_1)$  is

$$y_1 y = 2 p (x + x_1).$$

II. The slope of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  at the point  $(x_1, y_1)$  is

$$m = -\frac{b^2 x_1}{a^2 y_1}.$$

The equation of the tangent at  $(x_1, y_1)$  is

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1.$$

III. The slope of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  at the point  $(x_1, y_1)$  is

$$m = \frac{b^2 x_1}{a^2 y_1}.$$

The equation of the tangent at  $(x_1, y_1)$  is

$$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1.$$

IV. The slope of the hyperbola  $xy = k$  at the point  $(x_1, y_1)$  is

$$m = -\frac{y_1}{x_1}.$$

The equation of the tangent at  $(x_1, y_1)$  is

$$xy_1 + x_1 y = 2 k.$$

These formulas for tangents are easily remembered

if one notices that they can be obtained from the given equation of the curve by replacing

$$x^2 \text{ by } x_1x,$$

$$y^2 \text{ by } y_1y,$$

$$x \text{ by } \frac{x + x_1}{2},$$

$$y \text{ by } \frac{y + y_1}{2},$$

$$,, \text{ by } \frac{xy_1 + x_1y}{2}$$

the constant being left unchanged.

### Exercises

Find the equation of the tangents to the following curves at the points indicated.

$$1. x^2 + 4y^2 = 20, \quad (2, 2). \quad 2. y^2 = 8x, \quad (2, 4).$$

$$3. x^2 = -4y, \quad (2, -1). \quad 4. x^2 - y^2 = 5, \quad (3, 2).$$

$$5. ax^2 + by^2 = c, \quad (x_1, y_1). \quad 6. Ax^2 - By = 0, \quad (x_1, y_1).$$

$$7. Ax^2 - By^2 = C, \quad (x_1, y_1). \quad 8. xy = 8, \quad (4, 2).$$

**71. Normals.** The normal at any point of a curve is the line which is perpendicular to the tangent at that point.

The equation of the normal can always be found by writing the equation of the tangent and then writing

the equation of the line perpendicular to the tangent passing through the point of contact.

*Example.* Find the equation of the normal to  $y^2 = 4px$  at the point  $(x_1, y_1)$ .

Solution: The equation of the tangent is

$$y_1 y = 2p(x + x_1).$$

$$\therefore \text{Slope of tangent} = \frac{2p}{y_1}.$$

$$\text{The equation of the normal is } y - y_1 = \frac{y_1}{2p}(x - x_1).$$

### Exercises

Find the equations of the normals to:

1.  $y^2 = 8x$  at  $(8, 8)$ .
2.  $y^2 - 2x^2 + 14 = 0$  at  $(3, -2)$ .
3.  $16x^2 + 25y^2 = 400$  at  $(3, \frac{16}{5})$ .
4.  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  at  $(x_1, y_1)$ .
5.  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  at  $(x_1, y_1)$ .

**72. Subtangents; subnormals.** The lengths of the projections on the  $x$ -axis of the segments of the tangent and normal included between the point of contact and the point of intersection with the  $x$ -axis, are called the **subtangent** and the **subnormal**.

Prove for an ellipse, at the point  $(x_1, y_1)$

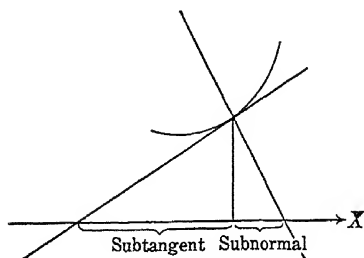


FIG. 92

$$\text{subtangent} = \left| \frac{a^2 - x_1^2}{x_1} \right|$$

$$\text{subnormal} = \left| \frac{b^2}{a^2} x_1 \right|.$$

For a hyperbola

$$\text{subtangent} = \left| \frac{a^2 - x_1^2}{x_1} \right|$$

$$\text{subnormal} = \left| \frac{b^2}{a^2} x_1 \right|.$$

For a parabola

$$\text{subtangent} = |2 x_1|,$$

$$\text{subnormal} = |2 p|.$$

**73. The tangent to a conic in terms of its slope.** The equation of any line with slope  $m$  may be written

$$y = mx + k.$$

We now wish to determine the values of  $k$  for which

the line will be tangent to the given conic. Every line of the system will in general cut the conic in two points. If the points are coincident, then the line is tangent. Therefore we must solve the equation of the line with the equation of the conic and impose upon the solution the condition for equal roots. (See § 2.)

*Example 1.* Find in terms of the slope  $m$  the equations of the tangents to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution: The equation of any line with slope  $m$  is

$$y = mx + k.$$

Solving with the given equation we have,

$$b^2x^2 + a^2(mx + k)^2 = a^2b^2,$$

or

$$(b^2 + a^2m^2)x^2 + 2a^2mkx + (a^2k^2 - a^2b^2) = 0.$$

The condition for equal roots is that the discriminant be equal to zero. § 2.

Therefore;

$$4a^4m^2k^2 - 4(b^2 + a^2m^2)(a^2k^2 - a^2b^2) = 0,$$

which when simplified and solved for  $k$ , gives,

$$k = \pm \sqrt{a^2m^2 + b^2}.$$

∴ The equations of the tangents are

$$y = mx \pm \sqrt{a^2m^2 + b^2}.$$

If  $a = b = r$ , the ellipse is a circle of radius  $r$ . Hence

the equations of the tangents to the circle  $x^2 + y^2 = r^2$  are

$$y = mx \pm r \sqrt{m^2 + 1}.$$

It is left as an exercise to prove that the equations of the tangents with slope  $m$  to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  are

$$y = mx \pm \sqrt{a^2 m^2 - b^2},$$

and to the parabola  $y^2 = 4px$ ,

$$y = mx + \frac{p}{m}.$$

*Example 2.* Find the equations of the tangents to  $4x^2 + 9y^2 = 36$  parallel to  $3x - y = 7$ .

Solution: The given equation is  $\frac{x^2}{9} + \frac{y^2}{4} = 1$  and  $a = 3, b = 2$ . The slope of the given line, and therefore the slope of the required line, is  $m = 3$ . Hence the equations of the tangents are from Ex. 1

$$y = 3x \pm \sqrt{(3)^2(3)^2 + (2)^2},$$

or  $y = 3x \pm \sqrt{85}.$

*Example 3.* Find the equations of the tangents from  $(7, 1)$  to the circle  $x^2 + y^2 = 25$ .

Solution: The equations of the tangents to the circle in terms of slope  $m$  are

$$y = mx \pm 5 \sqrt{m^2 + 1}.$$

Since the tangents are to pass through the point  $(7, 1)$ ,

the equations are satisfied by  $x = 7$ ,  $y = 1$ . Therefore

$$1 = 7m \pm 5\sqrt{m^2 + 1},$$

or 
$$1 - 7m = \pm 5\sqrt{m^2 + 1}$$

Squaring and simplifying we have,

$$12m^2 - 7m - 12 = 0,$$

or 
$$m = \frac{4}{3}, \quad -\frac{3}{4}.$$

$\therefore$  The equations of the required tangents are

$$y - 1 = \frac{4}{3}(x - 7), \quad \text{or } 4x - 3y - 25 = 0,$$

$$y - 1 = -\frac{3}{4}(x - 7), \quad \text{or } 3x + 4y - 25 = 0.$$

### Exercises

Find the equations of the tangents to the following curves, satisfying the conditions stated:

1.  $y^2 = 4x$ , slope = 1.
2.  $y^2 = 8x$ , parallel to  $2x - y = 0$ .
3.  $y^2 = -4x$ , inclined at  $\theta = 135^\circ$ .
4.  $y^2 = 16x$ , perpendicular to  $2x + y + 7 = 0$ .
5.  $x^2 + y^2 = 13$  from  $(1, 5)$ .
6.  $y^2 = 4x$  from  $(-1, 0)$ .
7.  $x^2 + y^2 = 5$  from  $(3, 1)$ .
8.  $y^2 = 16x$ , from  $(2, -6)$ .
9.  $y^2 = -4x$ , from  $(3, -2)$ .
10.  $y^2 = 2x$ , from  $(-4, 1)$ .
11.  $\frac{x^2}{25} - \frac{y^2}{16}$ : 1 parallel to  $x - y = 0$ .

12.  $9x^2 - y^2 = 9$  perpendicular to  $x - 5y + 5 = 0$ .

13.  $x^2 + y^2 = 20$  from  $(6, 2)$ .

14.  $y^2 = 4x$  at  $(1, -2)$ .

15.  $x^2 - 4y^2 = 21$  at  $(5, 1)$ .

16.  $9x^2 - y^2 = 5$  at points whose abscissas are 1.

**74. Construction of a tangent to a parabola.** In order to construct a straight line with a ruler, it is first necessary to locate two points on the line which will in turn fix the position of the ruler and hence permit the drawing of the line. Consequently, in order to construct the tangent to a parabola we must have some way of locating a point on the tangent in addition to the point of tangency.

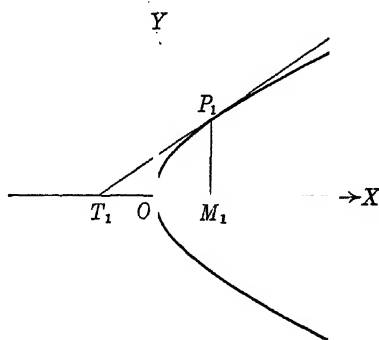


FIG. 93

The tangent  $y_1y = 2p(x + x_1)$  to the parabola  $y^2 = 4px$  at the point  $P_1(x_1, y_1)$  meets the  $x$ -axis at the point  $T_1(-x_1, 0)$ . This shows that  $T_1O = OM_1$ , where  $M_1$  is the foot of the perpendicular dropped from  $P_1$  on the axis of the curve. We have here a purely geometric

property of the parabola, *i.e.*, a property which is not dependent on the coordinate axes:

If the tangent to a parabola at  $P_1$  meets the axis of the curve in  $T_1$  and the foot of the perpendicular to the axis through  $P_1$  meets the axis in  $M_1$ , the vertex of the curve bisects the segment  $T_1M_1$ .

This leads to a method for constructing the tangent to a parabola at a given point  $P_1$  if the axis and vertex of the parabola are given. Let the student explain in detail.

**75. The focal property of a parabola.** Let  $F$  be the focus of the parabola,  $D_1$  the point where the directrix crosses the axis, and  $L_1$  the point where the line through  $P_1$  parallel to the axis meets the directrix. From the definition of the parabola, we have  $FP_1 = L_1P_1 = D_1M_1$ . But  $T_1O = OM_1$ . Since  $D_1O = OF$ , we have  $T_1D_1 = FM_1$ , and therefore,

$$T_1F = D_1M_1 = L_1P_1 = FP_1.$$

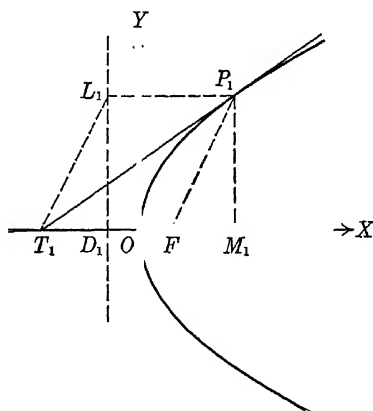


FIG. 94

Hence the quadrilateral  $T_1FP_1L_1$  is a rhombus. Since a diagonal of a rhombus bisects the angles to which it is drawn, the tangent at any point  $P_1$  of a parabola bisects the angle between the focal radius  $FP_1$  drawn to  $P_1$  and the line through  $P_1$  parallel to the axis, i.e.  $\angle^1 = \angle^2$ . Since  $\angle^2 = \angle^3$ , it follows that  $\angle^1 = \angle^3$ .

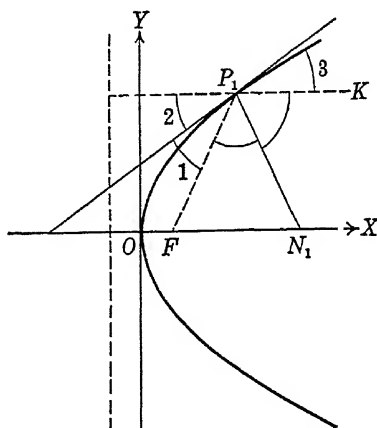


FIG. 95

The normal  $P_1N_1$  drawn to the parabola at  $P_1$  bisects the supplementary angle  $FP_1K$  formed by the focal radius and the line through  $P_1$  parallel to the axis. If the parabola is revolved about its axis, a surface is generated, which if mirrored will have the property that *all* light rays issuing from  $F$  are reflected by the surface in the direction of the axis. We know from experimenting with light that when a light ray strikes a mirror the angle of incidence equals the angle of reflection, i.e., the angle at which it strikes equals the angle at which it is reflected. Hence from the property which we have just

proved for the parabola, a light ray issuing from  $F$  and striking at  $P_1$  will be reflected along  $P_1K$ . This is the principle underlying the parabolic reflectors familiar in searchlights.

**76. Construction of a tangent to an ellipse. Auxiliary circles.** The tangent  $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$  to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  meets the  $x$ -axis at  $T_1\left(\frac{a^2}{x_1}, 0\right)$ , a point depending only on  $x_1$  and  $a$ . In other words, the tangents to all ellipses,  $b^2x^2 + a^2y^2 = a^2b^2$ , at points with given

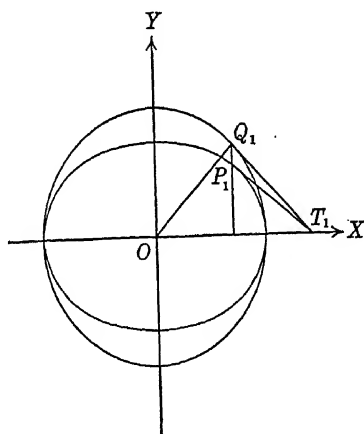


FIG. 96

abscissa  $x_1$  and in which  $a$  is *constant*, meet the  $x$ -axis in the same point. One of these ellipses is the circle with radius  $a$ . Hence to construct the tangent to an ellipse at a given point  $P_1$ , we need only draw the circle with major axis as diameter, draw a line through  $P_1$

perpendicular to this axis, meeting the circle in  $Q_1$ ; draw the tangent to the circle at this point and find the intersection  $T_1$  of this tangent with the major axis produced; the line  $P_1T_1$  is then the desired tangent to the ellipse.

The circle drawn on the major axis of an ellipse as diameter is called the **major auxiliary circle** of the ellipse; similarly, the circle drawn on the minor axis of the ellipse as diameter is called the **minor auxiliary circle**. The minor circle can be used equally well for the construction of the tangent. Let the student explain.

**77. The focal property of an ellipse.** We shall now prove the following remarkable property of the ellipse.

The normal at any point  $P_1$  of an ellipse bisects the angle formed by the focal radii drawn from  $P_1$ .

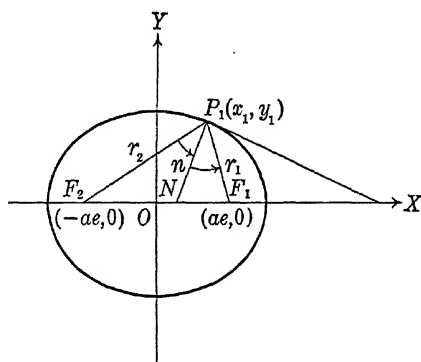


FIG. 97

Denote by  $r_2$ ,  $r_1$  and  $n$  the focal radii and the normal drawn from the point  $P_1(x_1, y_1)$  on the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$ .

We wish to prove that  $\angle (r_2n) = \angle (nr_1)$ . There are several ways of proving the theorem. The proof below is not the shortest, but it is direct and offers a good example of how a relatively complicated algebraic expression will often simplify.

Since the coordinates of  $F_2$  are  $(-ae, 0)$ , the slope of  $r_2$  is  $\frac{y_1}{x_1 + ae}$ ; from §§ 70, 71 the slope of  $n$  is  $\frac{a^2y_1}{b^2x_1}$ . Hence (§ 37),

$$\begin{aligned}\tan (r_2n) &= \frac{\frac{a^2y_1}{b^2x_1} - \frac{y_1}{x_1 + ae}}{1 + \frac{a^2y_1^2}{b^2x_1(x_1 + ae)}} = \frac{y_1(a^2x_1 + a^3e - b^2x_1)}{b^2x_1^2 + a^2y_1^2 + b^2aex_1} \\ &= \frac{y_1(a^2e^2x_1 + a^3e)}{a^2b^2 + b^2aex_1} = \frac{y_1 a^2e (ex_1 + a)}{b^2a (a + ex_1)} = \frac{aey_1}{b^2}.\end{aligned}$$

In similar fashion we could calculate  $\tan (nr_1)$ ; but since  $r_1$  is obtained from  $r_2$  by simply changing the sign of  $ae$ , we can conclude at once that  $\tan (r_1n) = -\frac{aey_1}{b^2}$  and hence that  $\tan (nr_1) = \frac{aey_1}{b^2}$ . This proves the theorem.

### MISCELLANEOUS EXERCISES

1. Find the equations of the tangents to the ellipse  $4x^2 + 5y^2 = 20$  which make an angle of  $45^\circ$  with the  $x$ -axis.
2. Find the equations of the tangents to the ellipse  $x^2 + 3y^2 = 3$  which are perpendicular to the line  $x - y + 1 = 0$ .
3. Find the points of intersection of the hyperbola  $25x^2 - 9y^2 = 225$  and the straight line  $25x + 12y = 0$ .

4. Prove that the equation of the tangent to the ellipse  $Ax^2 + By^2 = C$  at the point  $(x_1, y_1)$  is  $Ax_1x + By_1y = C$ .
5. Find the angle at which the line  $y = x$  cuts the hyperbola  $4x^2 - y^2 = 15$ .
6. Find the equations of the tangent and the normal to the ellipse  $4x^2 + y^2 = 5$  at the point  $(1, -1)$ .
7. Find the equations of the tangents from the point  $(1, -1)$  to the hyperbola  $x^2 - y^2 = 5$ .
8. What is the area of the triangle formed by the coordinate axes and the tangent to the ellipse  $5x^2 + 2y^2 = 13$  at the point  $(1, 2)$ ?
9. Find the equations of the tangents from the point  $(3, 2)$  to the hyperbola  $x^2 - 2y^2 = 5$ .
10. Find the equations of the tangents and normals to the ellipse  $5x^2 + 3y^2 = 137$  at the points whose ordinate is 2.
11. Find the equation of the chord of the hyperbola  $25x^2 - 16y^2 = 400$  which is bisected by the point  $(5, 3)$ .
12. Find the equation of the normal to the hyperbola  $16x^2 - 9y^2 = 144$  at the extremity of the latus rectum which lies in the first quadrant.
13. Prove that the slopes of the tangents at the extremities of a latus rectum of the hyperbola  $b^2x^2 - a^2y^2 = a^2b^2$  are  $\pm e$  where  $e$  is the eccentricity.
14. Prove that the point  $(a \sec \phi, b \tan \phi)$  lies on the hyperbola  $b^2x^2 - a^2y^2 = a^2b^2$ , for all values of  $\phi$ .
15. Prove that the hyperbola  $b^2x^2 - a^2y^2 = a^2b^2$  has no tangents whose slopes lie between  $b/a$  and  $-b/a$ .
16. Find the equations of the tangents and normals to the ellipse  $9x^2 + 16y^2 = 144$  at the extremities of a latus rectum.

17. Find the equations of the tangents to the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  at the extremities of a latus rectum and prove that they meet on the corresponding directrix.
18. Find the equations of the tangents to the ellipse  $x^2 + 4y^2 = 20$  from the point  $(6, 1)$ .
19. Prove that the tangents drawn to the ellipse  $x^2 + 3y^2 = 12$  from the point  $(0, 4)$  are perpendicular to each other.
20. Find the points on the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  such that the tangents drawn at these points have equal intercepts on the axes. Prove also that the length of the perpendicular from the center to either of these tangents is  $\frac{\sqrt{2(a^2 + b^2)}}{2}$ .
21. If  $e$  and  $e'$  are the eccentricities of two conjugate hyperbolas, prove that:  $\frac{1}{e^2} + \frac{1}{e'^2} = 1$ .
22. Prove that the foci of a hyperbola and those of its conjugate hyperbola lie on a circle.
23. What is the eccentricity of a hyperbola in which the vertices bisect the segments from the center to the foci?
24. Prove that in a rectangular hyperbola the product of the focal distances of any point  $P$  is equal to the square of the distance of  $P_1$  from the center.
25. If the normal at any point  $P$  of a rectangular hyperbola meets the axes in  $N_1, N_2$ , prove that  $PN_1 = PN_2 = PC$ , where  $C$  is the center.
26. Prove that the segment of any tangent to a hyperbola between the point of contact and the directrix subtends a right angle at the corresponding focus. Show how this property can be used to construct the tangent at a given point, if a focus and the corresponding directrix are given.

27. Prove that the line joining a point  $P$  on a hyperbola to the center and the line through a focus perpendicular to the tangent at  $P$  meet on a directrix.
28. Prove that the vertices of a hyperbola subtend a right angle at an extremity of the conjugate axis, if and only if the hyperbola is rectangular.
29. Prove that if any tangent to a hyperbola meets the tangents at the vertices in  $M$  and  $N$ , then  $M, N$ , and the two foci lie on a circle.
30. Prove that any ellipse and hyperbola which have the same foci cut orthogonally.
31. Given the system of curves

$$\frac{x^2}{a^2 + k} + \frac{y^2}{b^2 + k} = 1,$$

where  $a \neq b$ . For what values of  $k$  does the equation represent an ellipse, and for what values does it represent a hyperbola? Show that all the ellipses and hyperbolas of this system of curves have the same foci.

## CHAPTER VIII

### TRANSFORMATION OF COORDINATES

**78. Transformations.** The coordinates of a point are fixed if the point is referred to a given set of axes. If the axes are changed, the coordinates of the point are changed. Similarly the equation of a given curve is changed if the axes of reference are changed.

The operation of changing the axes is called transformation of coordinates. We shall consider two types of transformations. If the new axes are respectively parallel to the old axes and pass through a new origin, the transformation is called *translation of axes*. If the origin is unchanged, but the axes are rotated through a given angle, the transformation is called *rotation of axes*.

**79. Translation of axes.** Suppose the axes  $OX$  and  $OY$  are translated so the new origin is at  $O'$ . If the coordinates of  $O'$ , with respect to the original axes are

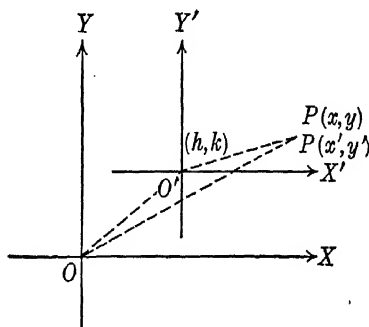


FIG. 98

$(h, k)$ , and the coordinates of any point  $P$  before and after translation are  $(x, y)$  and  $(x', y')$  respectively, then since

$$\text{Proj}_x OP = \text{Proj}_x OO' + \text{Proj}_x O'P,$$

$$x = h + x'.$$

Similarly,  $y = k + y'.$

*∴ Theorem 1.* If  $x$  and  $y$  are the coordinates of any point before translation to a new origin  $(h, k)$  and  $x'$  and  $y'$  are the coordinates of the same point after translation, then

$$\begin{aligned} x &= x' + h \\ (1) \quad y &= y' + k. \end{aligned}$$

*Example 1.* Find the coordinates of the point  $(4, -5)$  if the axes are translated to  $(-2, 3)$ .

Solution:  $x = 4, y = -5, h = -2, k = 3.$

$$\therefore 4 = x' - 2, \quad -5 = y' + 3,$$

$$\text{and } x' = 6, \quad y' = -8.$$

*Example 2.* Transform the equation

$$4x^2 + 9y^2 - 8x - 36y + 4 = 0,$$

to axes parallel to the original axes, so that in the new equation there shall be no terms in  $x$  and  $y$ .

Method I. Solution: The formulas of transformation are

$$x = x' + h, \quad y = y' + k,$$

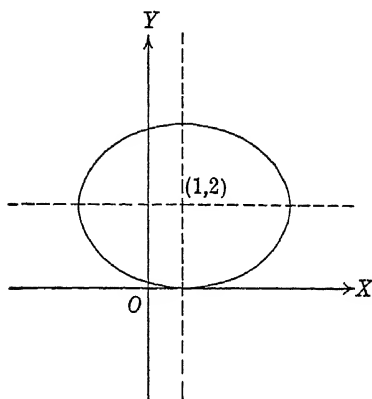


FIG. 99

where the values of  $h$  and  $k$  are to be determined. The given equation becomes

$$4(x' + h)^2 + 9(y' + k)^2 - 8(x' + h) - 36(y' + k) + 4 = 0.$$

Expanding and collecting terms we have

$$4x'^2 + 9y'^2 + x'(8h - 8) + y'(18k - 36) + (4h^2 + 9k^2 - 8h - 36k + 4) = 0.$$

By the conditions of the problem

$$8h - 8 = 0, \quad 18k - 36 = 0,$$

from which it follows that  $h = 1$ ,  $k = 2$ .

Therefore the coordinates of the new origin are (1, 2) and the new equation is

$$4x'^2 + 9y'^2 = 36,$$

$$\text{or} \quad \frac{x'^2}{9} + \frac{y'^2}{4} = 1.$$

Method II. The given equation may be written in the form

$$4(x^2 - 2x) + 9(y^2 - 4y) = -4.$$

Completing the squares of the terms in parentheses and remembering that in order to balance the equation we must add to the right-hand member the same quantity we added to the left, we have

$$4(x^2 - 2x + 1) + 9(y^2 - 4y + 4) = -4 + 4 + 36,$$

$$4(x - 1)^2 + 9(y - 2)^2 = 36,$$

$$\frac{(x - 1)^2}{9} + \frac{(y - 2)^2}{4} = 1.$$

Placing  $x' = x - 1$ ,  $y' = y - 2$ , our new equation is

$$\frac{x'^2}{9} + \frac{y'^2}{4} = 1.$$

*Example 3.* Find the coordinates of the center, the vertices, the foci, and the equations of the directrices of the ellipse,

$$25x^2 + 9y^2 - 50x + 36y - 164 = 0.$$

$$\text{Solution: } 25(x^2 - 2x) + 9(y^2 + 4y) = 164,$$

$$25(x^2 - 2x + 1) + 9(y^2 + 4y + 4) = 164 + 25 + 36,$$

$$25(x - 1)^2 + 9(y + 2)^2 = 225,$$

or 
$$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{25} = 1.$$

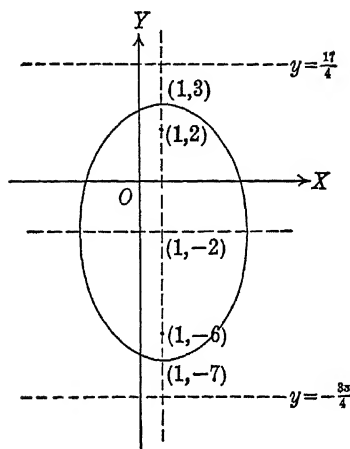


FIG. 100

We can now conclude that the center is at  $(1, -2)$  and that the major axis is parallel to the  $y$ -axis. Moreover,  $a = 5$ ,  $b = 3$ ,  $c = 4$ ,  $e = \frac{4}{5}$ , and  $\frac{a}{e} = \frac{25}{4}$ . Sketching the ellipse we find from the figure that the coordinates of the vertices are  $(1, 3)$  and  $(1, -7)$ , and the coordinates of the foci are  $(1, 2)$  and  $(1, -6)$ . The equations of the directrices are  $y = \frac{17}{4}$  and  $y = -\frac{33}{4}$ .

**Theorem 2.** *If the equation*

$$Ax^2 + By^2 + Dx + Ey + C = 0$$

*has a locus it is an ellipse, a hyperbola, a parabola, two straight lines or a single point.*

Outline of proof: Assume none of the coefficients is zero. Then  $A$  can be taken positive (why?) and the equation can be written

$$(2) \quad A \left( x + \frac{D}{2A} \right)^2 + B \left( y + \frac{E}{2B} \right)^2 = \frac{D^2}{4A} + \frac{E^2}{4B} - C.$$

Denote the right-hand member by  $R$ . The following cases arise.

$$\text{If } B < 0 \text{ and } \begin{cases} R < 0 \\ R = 0 \\ R > 0 \end{cases} \text{ we have } \begin{cases} \text{a hyperbola.} \\ \text{two lines.} \\ \text{a hyperbola.} \end{cases}$$

$$\text{If } B > 0 \text{ and } \begin{cases} R < 0 \\ R = 0 \\ R > 0 \end{cases} \text{ we have } \begin{cases} \text{no locus.} \\ \text{a point.} \\ \text{an ellipse.} \end{cases}$$

If  $A$  or  $B$  is zero the method fails. If  $A = 0$  and  $B \neq 0$ ,  $D \neq 0$ , the equation can be written

$$B \left( y + \frac{E}{2B} \right) = -D \left( x - \frac{E^2 - 4BC}{4BD} \right)$$

which is a parabola. Now consider the cases

$$A \neq 0, B = 0, E \neq 0; \quad A = 0, D = 0; \quad B = 0, E = 0.$$

### Exercises

1. What are the new coordinates of the points  $(2, 1)$ ,  $(-3, 4)$ ,  $(5, -5)$ ,  $(0, 0)$  if the origin is transferred to the point  $(3, -2)$  the axes being parallel to the old axes?
2. Transform the equation  $x^2 + 4x + y^2 - 8y + 1 = 0$  referred to new axes parallel to the old axes and meeting at the point  $(-2, 4)$ .

3. Transform the equation  $-x^2 + y^2 + 4x = 0$  referred to new axes parallel to the old axes, and meeting at the point  $(2, 0)$ .
4. Transform the equation  $y^2 - 8x + 4y - 4 = 0$  referred to new axes parallel to the old axes and meeting at the point  $(4, -5)$ .
5. Prove that  $xy + 2x - y - 7 = 0$  can be reduced to the form  $xy = k$  by translation of axes.

For each of the following parabolas determine the coordinates of the vertex, the coordinates of the focus, the equation of the directrix, and sketch the curve.

6.  $y^2 - 8y - 2x + 18 = 0$ .
7.  $y^2 + 2y - 4x + 5 = 0$ .
8.  $x^2 + 6x - 5y - 16 = 0$ .
9.  $x^2 - 8x + y + 15 = 0$ .
10.  $y^2 - 4x + 4y = 0$ .
11.  $2y^2 - 8y - 3x + 11 = 0$ .

For each of the following loci determine the coordinates of the center, the vertices, the foci, the equation of the directrices, equations of asymptotes when they exist, and sketch the curve.

12.  $4x^2 + 9y^2 - 8x - 18y - 23 = 0$ .
13.  $4x^2 - 8x + y^2 + 2y + 1 = 0$ .
14.  $25x^2 + 9y^2 - 50x + 36y - 164 = 0$ .
15.  $16x^2 - 9y^2 + 32x + 54y - 209 = 0$ .
16.  $4x^2 + y^2 - 8x + 4y + 4 = 0$ .
17.  $9x^2 - 16y^2 + 18x - 96y - 279 = 0$ .
18.  $9x^2 + 4y^2 + 36x - 16y + 16 = 0$ .
19.  $x^2 - y^2 + 2x - 2y - 2 = 0$ .
20.  $2x^2 - 4y^2 + 12x + 16y - 7 = 0$ .

**80. Rotation of axes.** Let us rotate the axes  $OX$  and  $OY$  about the origin through the angle  $\theta$  into the positions  $OX'$ ,  $OY'$ . These two lines we shall use as a new set of coordinate axes.

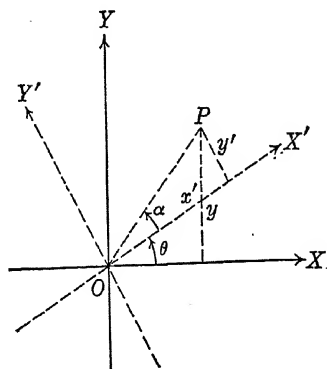


FIG. 101

If  $x$  and  $y$  are the original coordinates of a point  $P$ ,  $x'$  and  $y'$  the new coordinates, and  $\theta$  the angle of rotation,

$$(2) \quad \begin{aligned} x &= x' \cos \theta - y' \sin \theta, \\ y &= x' \sin \theta + y' \cos \theta. \end{aligned}$$

These equations can be derived in a variety of ways. One method is to notice that

$$\begin{aligned} \frac{y}{OP} &= \sin(\theta + \alpha) \\ &= \sin \theta \cos \alpha + \cos \theta \sin \alpha. \end{aligned}$$

But  $\sin \alpha = \frac{y'}{OP}$  and  $\cos \alpha = \frac{x'}{OP}$ .

Substituting we get the second equation of (2). The first equation is obtained in precisely the same manner.

If we solve formulas (2) for  $x'$  and  $y'$ , we get

$$(3) \quad \begin{aligned} x' &= x \cos \theta + y \sin \theta, \\ y' &= -x \sin \theta + y \cos \theta. \end{aligned}$$

*Example.* Find the equation of the locus of  $xy = 4$  when referred to new axes obtained from the given axes by rotating them through a positive angle of  $45^\circ$ .

Solution: Here  $\theta = 45^\circ$ . Hence the equations of transformation are

$$\begin{aligned} x &= \frac{1}{\sqrt{2}} x' - \frac{1}{\sqrt{2}} y', \\ y &= \frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{2}} y'. \end{aligned}$$

Substituting we get

$$\frac{1}{2} x'^2 - \frac{1}{2} y'^2 = 4 \text{ or } x'^2 - y'^2 = 8,$$

which is a rectangular hyperbola.

### Exercises

1. Find the coordinates of the points  $(3, 2)$ ,  $(4, -5)$ ,  $(0, 2)$  after the axes have been rotated through  $30^\circ$ ; through  $90^\circ$ .
2. Transform the equation  $x^2 + y^2 = 25$  by rotating the axes through  $60^\circ$ .
3. Transform the equation  $x^2 + y^2 = r^2$  by rotating the axes through an angle  $\theta$  and show that the form of the equation is left unchanged.

4. Transform the equation  $xy = k$  by rotating the axes through an angle of  $45^\circ$ .
5. Transform the equation  $y + x = 0$  by rotating the axes through an angle of  $135^\circ$ .
6. Transform the equation  $xy = y - 2x + 6$  by translating the axes so the new origin is at  $(1, -2)$  and then rotate the axes through  $45^\circ$ .
7. Transform  $8x^2 - 4xy + 5y^2 - 36 = 0$  by rotating the axes through an angle  $\theta$  where  $\tan \theta = 2$ .
8. Transform  $4x^2 + 15xy - 4y^2 - 20 = 0$  by rotating the axes through an angle  $\theta$  such that  $\theta = \tan^{-1} \frac{3}{5}$ .
9. Transform  $x^2 + 4xy + y^2 = 10$  by rotating the axes through an angle  $\theta$  of  $45^\circ$ .

**81. Theorem.** *The equation  $Ax^2 + Fxy + By^2 + Dx + Ey + C = 0$  has for its locus an ellipse, a hyperbola, a parabola, two straight lines (which may coincide), a single point or no locus.*

**Analysis:** To prove this theorem we shall show that if the axes are rotated through a properly chosen angle  $\theta$ , the given equation can be reduced to the form

$$A'x^2 + B'y^2 + D'x + E'y + C = 0. \quad \text{See Theorem 2, § 79.}$$

**Solution:** If we rotate the axes through an angle  $\theta$  by means of formulas (2), the given equation becomes

$$\begin{aligned} & A(x' \cos \theta - y' \sin \theta)^2 + F(x' \cos \theta - y' \sin \theta) \\ & \quad (x' \sin \theta + y' \cos \theta) + B(x' \sin \theta + y' \cos \theta)^2 \\ & + D(x' \cos \theta - y' \sin \theta) + E(x' \sin \theta + y' \cos \theta) \\ & + C = 0. \end{aligned}$$

Expanding and collecting terms we have,

$$\begin{aligned} & [A \cos^2 \theta + F \sin \theta \cos \theta + B \sin^2 \theta] x'^2 \\ & + [-2 A \sin \theta \cos \theta + F \cos^2 \theta - F \sin^2 \theta \\ & + 2 B \sin \theta \cos \theta] x'y' + [A \sin^2 \theta - F \sin \theta \cos \theta \\ & + B \cos^2 \theta] y'^2 + [D \cos \theta + E \sin \theta] x' \\ & + [-D \sin \theta + E \cos \theta] y' + C = 0. \end{aligned}$$

We wish to choose  $\theta$ , so that the coefficient of  $x'y'$  is zero.

Hence it is necessary that

$$F (\cos^2 \theta - \sin^2 \theta) + (2 B - 2 A) \sin \theta \cos \theta = 0,$$

$$(4) \quad \text{or} \quad F \cos 2 \theta + (B - A) \sin 2 \theta = 0.$$

If  $A$  is not equal to  $B$  this can be written in the more convenient form.

$$(5) \quad \tan 2 \theta = \frac{F}{A - B}. \quad (A \neq B).$$

If  $F \neq 0$  we can write it in the form

$$\cot 2 \theta = \frac{A - B}{F} \quad (F \neq 0).$$

There are an infinite number of values of  $\theta$  which satisfy this equation; a single one of these values will reduce the equation so that it contains no  $xy$  term.

There is just one positive angle for  $\theta$  between  $0^\circ$  and  $180^\circ$  and it is customary to choose this angle.

*Example.* Determine through what angle the axes must be rotated in order to remove the  $xy$  term in the equation  $8x^2 + 4xy + 5y^2 - 36 = 0$ . Sketch the locus.

Solution:  $A = 8$ ,  $F = 4$ ,  $B = 5$ .

$$\tan 2 \theta = \frac{F}{A - B} = \frac{4}{8 - 5} = \frac{4}{3}.$$

But 
$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

Hence 
$$\frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{4}{3} \text{ or } 2 \tan^2 \theta + 3 \tan \theta - 2 = 0.$$

$$\tan \theta = \frac{1}{2}, -2.$$

We choose  $\tan \theta = \frac{1}{2} \therefore \sin \theta = \frac{1}{\sqrt{5}}, \cos \theta = \frac{2}{\sqrt{5}}.$

Substituting  $x = \frac{2}{\sqrt{5}} x' - \frac{1}{\sqrt{5}} y', y = \frac{1}{\sqrt{5}} x' + \frac{2}{\sqrt{5}} y',$

in the given equation and simplifying, we have

$$9x'^2 + 4y'^2 = 36.$$

The desired graph is obtained by plotting the curve with respect to the new axes.

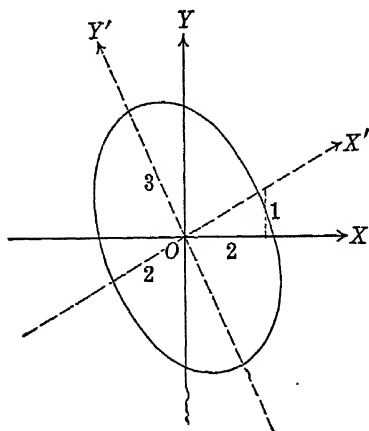


FIG. 101a

## Exercises

Determine the angle  $\theta$  through which the loci of the following equations must be rotated in order that their new equations shall contain no  $xy$  term. Determine in each case the new equation and use it to draw the locus of the original equation.

1.  $x^2 + 2xy + y^2 - 12 = 0$ .
2.  $x^2 + 4xy + 4y^2 = 36y$ .
3.  $4x^2 - 4xy + y^2 - 6 = 0$ .
4.  $2x^2 - 12xy - 3y^2 + 42 = 0$ .
5.  $x^2 + xy + y^2 - 7x + y = 4$ .
6.  $x^2 - 2\sqrt{3}xy + 3y^2 - 4\sqrt{3}x = 0$ .
7.  $x^2 + 2xy + y^2 - 3x = 4$ .
8.  $x^2 + 4xy + y^2 - 6x + 7y = 8$ .
9.  $6x^2 + 4xy + 9y^2 + \sqrt{5}y + 2\sqrt{5}x = 10$ .
10. Prove that the locus of  $xy = c$  may be rotated about the origin so as to coincide with  $x^2 - y^2 = a^2$ , provided  $a^2 = \pm 2c$ .

**82. Conic through five points.** A circle can be passed through any three points not in a straight line. To find its equation we substitute the coordinates of the three points successively in  $x^2 + y^2 + ax + by + c = 0$ , and solve the resulting equations for  $a$ ,  $b$ , and  $c$ . Similarly, a parabola with vertical axis can be passed through three points not in a straight line. Its equation is of the form  $y = ax^2 + bx + c$ . We substitute and proceed as before. In general the number of points through which a locus can be passed is the same as the number of independent constants in its equation. Now the general equation of the second degree contains six constants

but these are not all independent, inasmuch as we can always divide all the terms by one of them, for certainly not all the constants are zero. For example, if in the general equation  $Ax^2 + Fxy + By^2 + Dx + Ey + C = 0$  we divide by  $A$ , assuming it is not zero, we write

$$x^2 + bxy + cy^2 + dx + ey + f = 0,$$

in which case there are only five constants. The equation of any conic that contains an  $x^2$  term can be written in this form. We can now proceed exactly as in the case of the circle, by substituting the coordinates of the five points successively and solving the resulting equations for  $b, c, d, e$ , and  $f$ .

### SIMPLE EXERCISES

Find the equation of the conic through:

1.  $(1, 1), (0, 0), (-3, 4), (-1, 2), (4, 0)$ .
2.  $(4, 2), (6, 3), (1, 0), (0, 1), (-1, 2)$ .
3.  $(0, 0), (-1, 2), (0, 1), (1, 2), (1, 1)$ .
4.  $(0, 3), (1, 0), (-1, -3), (2, 1), (3, -3)$ .
5. Prove that the equation of the conic through  $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5)$  is

$$\begin{array}{ccccccc} x^2 & xy & y^2 & x & y & 1 & \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 & \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 & \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 & \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 & \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 & \end{array} = 0.$$

[Hint: Proof similar to § 46.]

## CHAPTER IX

### OTHER LOCI

**83.** In the past chapters we have found the equations of certain simple loci. For example, in § 29, we found the locus of a point such that the slope of the line connecting it with a given fixed point was constant; in § 52, we found the locus of a point such that it was equally distant from a given fixed point and a given line not passing through the point; in § 16 and § 56, we found the locus of a point such that the sum of its distances from two fixed points was constant.

In these and other cases we simply expressed in algebraic language, *i.e.*, by an equation, the law governing the position of the given point whose locus we desired. This equation is satisfied by the coordinates of every point which satisfies the given conditions, and by no other points; see § 16.

#### Exercises

1. Find the equation of the locus of points which are equally distant from two fixed points. Describe the locus.
2. Find the equation of the locus of points at a given distance  $a$  from a fixed point. Describe the locus.
3. Find the equations of the locus of points which are equally distant from the coordinate axes.

4. A point moves so as to be three times as far from the point  $A$  as from the point  $B$ . Find the equation of its locus.
5. A point moves so that the sum of the squares of its distances from the vertices of a given square is constant. Find and describe the equation of its locus.
6. The distances of a moving point  $P$  from two fixed points are in the ratio  $p : q$ . Find the locus of  $P$ . What is the locus if  $p = q$ ?
7. Find the locus of a point which moves so that the sum of the squares of its distances from the sides or sides produced of a given square, is constant.
8. Find the locus of the center of a circle of radius  $r$  which passes through a fixed point  $(x_1, y_1)$ .
9. Find the locus of the center of a circle of radius  $r$  which touches the circle  $(x - h)^2 + (y - k)^2 = R^2$ , if  $R > r$ .
10. The base  $AB$  of  $\triangle ABC$  is fixed in length and position. Find the locus of the vertex  $C$ , if  $\angle C = 90^\circ$ .

**84. Auxiliary variables.** In many examples, it will be found that the work can be greatly simplified by introducing one or more new variables called auxiliary variables. In such cases if  $P$  is the point whose locus we desire, it is necessary that *we find one more equation connecting the auxiliary variables and the coordinates of  $P$ , than there are auxiliary variables.* From these equations, we must then eliminate the auxiliary variables. The resulting equation will contain only the coordinates of  $P$  and constants and will be the equation of a curve which, or some part of which, is the desired locus.

*Example 1.* A line of length  $AB$ , on which there is a fixed point  $P$  moves so its ends are always in two perpendicular lines. Find the locus of  $P$ .

Solution: Take the two  $\perp$  lines as coordinate axes. Let  $PA = a$ ,  $PB = b$ . Draw  $PM \perp OB$  and  $PS \perp OA$ . Then  $\angle SPA = \angle OBP = \theta$ . Since we have introduced an auxiliary variable  $\theta$ , we must find two equations connecting  $x$ ,  $y$ , and  $\theta$ . These equations are

$$\frac{x}{a} = \cos \theta, \quad \frac{y}{b} = \sin \theta$$

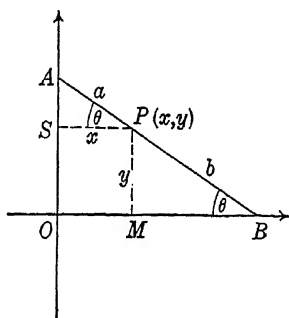


FIG. 102

Eliminating  $\theta$ , remembering  $\sin^2 \theta + \cos^2 \theta = 1$ , we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$\therefore$  The required locus is an ellipse whose center is the point of intersection of the two  $\perp$  lines and whose semi-axes are  $a$  and  $b$ .

*Example 2.* Find the equation of the locus of the mid-points of all lines joining any point  $P$  on the ellipse  $4x^2 + 9y^2 = 36$ , to the left-hand vertex.

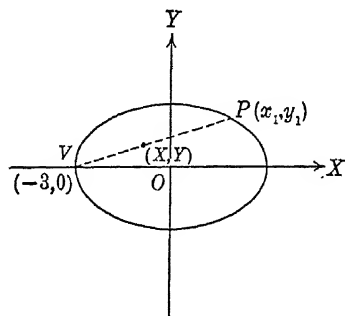


FIG. 103

**Solution:** The coordinates of the left-hand vertex are  $(-3, 0)$ . Let the coordinates of the variable point be  $(x_1, y_1)$  and the coordinates of the mid-point of  $VP$ ,  $(X, Y)$ .

Since we have introduced two auxiliary variables  $x_1$  and  $y_1$ , we must find three equations connecting these variables with the variables  $X$  and  $Y$  and then eliminate  $x_1$  and  $y_1$  from among the three equations.

From § 12 we have,

$$(1) \quad X = \frac{x_1 - 3}{2},$$

$$(2) \quad Y = \frac{y_1}{2}.$$

Since  $P$  is on the given ellipse we have

$$(3) \quad 4x_1^2 + 9y_1^2 = 36.$$

Solving (1) and (2) for  $x_1$  and  $y_1$  respectively, we have

$$x_1 = 2X + 3,$$

$$y_1 = 2Y.$$

Substituting these values in (3) we have,

$$4(2X + 3)^2 + 9(2Y)^2 = 36,$$

or  $4X^2 + 9Y^2 + 12X = 0$ , the equation of the desired locus. This is the equation of an ellipse, center at  $(-\frac{3}{2}, 0)$ , vertices at  $(-3, 0)$  and  $(0, 0)$ .

*Example 3.* Find the locus of the mid-points of the chords of the ellipse  $4x^2 + 9y^2 = 36$ , parallel to  $2x - y = 7$ .

Solution: The equation of any line parallel to the given line is  $y = 2x + k$ .

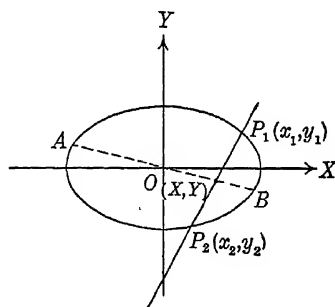


FIG. 104

To find the coordinates of the end points  $P_1$  and  $P_2$  we solve these equations simultaneously, giving

$$4x^2 + 9(2x + k)^2 = 36,$$

or

$$40x^2 + 36kx + (9k^2 - 36) = 0.$$

Let the coordinates of the mid-point of  $P_1 P_2$  be  $(X, Y)$ .

Then 
$$X = \frac{x_1 + x_2}{2}$$

But 
$$x_1 + x_2 = -\frac{36k}{40} = -\frac{9k}{10} \quad \S 2.$$

$\therefore$  
$$X = -\frac{9k}{20}.$$

But the point  $(X, Y)$  is on the line  $y = 2x + k$ .

$\therefore$  
$$Y = 2X + k.$$

Eliminating  $k$  we have

$$Y = 2X - \frac{20X}{2} \quad \text{as the equation of the locus,}$$

but the locus is not the entire line but merely the segment  $AB$ . Fig. 104.

### Exercises

1. Point  $P$  is not on line  $l$ . Find the locus of the mid-points of all the lines joining  $P$  to the points of  $l$ .
2. Find the locus of the mid-points of all chords drawn from a fixed point  $P$  on a circle.
3. The base of a triangle is fixed in length and position. Find the locus of the other vertex if one base angle is double the other.
4. Find the locus of the mid-points of all chords of a parabola which pass through the vertex.

5. The base  $AB$  of a triangle is fixed and the ratio of the sides  $AC$  and  $BC$  is constant. Find the locus of  $C$ .
6. Given three points  $A$ ,  $B$ , and  $C$ . If  $AB = 2a$  in length, find the locus of  $C$  if angle  $ACB$  is constant.
7. The base  $AB$  of a triangle  $ABC$  is fixed and  $\tan A = k \tan B$ ,  $k$  being a constant  $\neq 1$ . Find the locus of  $C$ .
8. The base  $AB$  of a triangle  $ABC$  is fixed and the length of the median drawn from  $A$  is constant. Find the locus of  $C$ .
9. Prove that the locus of the center of a circle which touches externally two given circles (of unequal radii) is a branch of a hyperbola.
10. The hypotenuse of a right-angled triangle slides with its extremities on two perpendicular lines. Find the locus of the vertex of the right angle.
11. Find the locus of the mid-points of all chords drawn through a fixed point within a circle.
12. Prove that the mid-points of any set of parallel chords of a parabola lie on a line parallel to the axis.
13. Find the locus of the mid-points of all chords of a hyperbola which have a slope  $m$ .
14. A variable chord  $PQ$  of an ellipse is perpendicular to the major axis.  $P$  is connected by a straight line with one vertex and  $Q$  with the other. Find the locus of the point of intersection of these lines.
15. A point  $P$  moves around on an ellipse. Perpendiculars are dropped from each vertex of the ellipse to the line connecting  $P$  to the other vertex. Find the locus of the point of intersection of these two perpendiculars.
16. A variable line is drawn parallel to the base  $AB$  of a fixed

triangle  $ABC$  meeting the side  $AC$  in  $D$  and the side  $CB$  in  $E$ . Find the locus of the point of intersection of  $AE$  and  $BE$ .

17. If  $M$  is the foot of the perpendicular dropped from a point  $P$  of a hyperbola on the transverse axis and if  $MP$  be produced to  $Q$  so that  $MQ$  is equal to either of the focal distances of  $P$ , prove that the locus of  $Q$  is one or the other of a pair of parallel straight lines.
18. If the slopes of the tangents drawn from a point  $P$  to the hyperbola  $a^2x^2 - a^2y^2 = a^2b^2$  are  $m_1$  and  $m_2$ , find the locus of  $P$  when  $m_1m_2 = c$  (a constant).
19.  $PP'$  is a double ordinate of an ellipse  $b^2x^2 + a^2y^2 = a^2b^2$ , and the normal at  $P$  meets  $OP'$  in  $Q$ . Show that the locus of  $Q$  is an ellipse.
20. Prove that the locus of points from which perpendicular tangents can be drawn to a parabola is the directrix of the parabola. [*Hint:* The slopes  $m_1$  and  $m_2$  of the tangents drawn from a point  $(x, y)$  to the parabola  $y^2 = 4px$  are the roots of the equation  $y = mx + \frac{p}{m}$  or  $xm^2 - ym + p = 0$ . Since the tangents are  $\perp$ ,  $m_1m_2 = -1$ .]
21. If a tangent to the parabola  $y^2 = 4px$  meets the axis in  $T$  and the tangent at the vertex  $O$  meets it in  $S$  and the rectangle  $TOSQ$  is completed, prove that the locus of  $Q$  is the parabola  $y^2 + px = 0$ .
22. Find the locus of the centers of circles which pass through a fixed point and cut a fixed straight line in chords of constant length.
23. The base  $AB$  of  $\triangle ABC$  is fixed in length and position, while vertex  $C$  moves on a line parallel to  $AB$ . Find the locus of the point of intersection of the medians.

24. The base  $AB$  of  $\triangle ABC$  is fixed in length and position while vertex  $C$  moves on a line parallel to  $AB$ . Find the locus of the point of intersection of the altitudes.
25. A variable tangent to a parabola meets the tangents at the extremities of the latus rectum in points  $P_1$  and  $P_2$ . Find the locus of the mid-point of the segment  $P_1P_2$ .
26. Tangents to a parabola meet at an angle of  $45^\circ$ . Find the locus of their point of intersection.
27.  $AB$  is a fixed chord of a circle and  $C$  is a moving point on the circle. Find the locus of the point of intersection of the medians of the triangle  $ABC$ .
28. A line revolves about a point  $A$  and meets a fixed circle in points  $P_1$  and  $P_2$ . Find the locus of  $P$  so situated on the line that

$$\overline{AP} = \overline{AP_1} - \overline{AP_2} - \overline{AP}.$$

29. Prove that the locus of a point from which perpendicular tangents can be drawn to the hyperbola  $a^2x^2 - a^2y^2 = a^2b^2$  is the circle  $x^2 + y^2 = a^2 - b^2$ . It is called the *director circle*. It reduces to a point (the center) when  $a = b$  and there is no circle when  $a < b$ .
30. Find the locus of the mid-points of the rectangles that can be inscribed in triangle  $ABC$ , if one side of the rectangle lies along  $AB$ .
31. In the parabola  $y^2 = 4px$ , find the locus of the mid-points of  
a) all ordinates;    b) all focal radii;    c) all focal chords.
32. In an ellipse, find the locus of the foot of a perpendicular dropped from a focus to a variable tangent.

33. Find the locus of the center of a circle that passes through the point whose coordinates are  $(0, 3)$ , and is tangent internally to the circle whose equation is  $x^2 + y^2 = 25$ .
34. A straight line of given length has its end points in two fixed perpendicular lines and forms with these lines a triangle of constant area  $k^2$ . Find the locus of the mid-point of this line.

**85. Diameters.** A straight line passing through the center of an ellipse or hyperbola is called a **diameter** of the conic. Every diameter of an ellipse meets the curve in two points; some of the diameters of a hyperbola meet the curve in two points. These points are then called the extremities of the diameter, and the distance between them is called the length of the diameter. Any line parallel to the axis of a parabola is called a **diameter** of the parabola.

*Theorem 1. The locus of the middle points of a set of parallel chords of an ellipse (hyperbola) is a diameter of the ellipse (hyperbola).*

If the equation of the ellipse is  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  and if  $m$  is the slope of the parallel chords, the equation of any one of them is

$$y = mx + k.$$

To find the coordinates of the end points  $P_1$  and  $P_2$  we solve these equations simultaneously, giving

$$b^2x^2 + a^2(mx + k)^2 = a^2b^2,$$

$$\text{or } x^2 (b^2 + a^2 m^2) + 2 a^2 m k x + (a^2 k^2 - a^2 b^2) = 0.$$

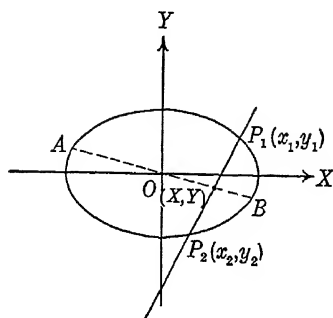


FIG. 105

If the coordinates of the mid-point of  $P_1P_2$  are  $(X, Y)$ ,

$$X = \frac{x_1 + x_2}{2}$$

$$\text{But, from § 2, } x_1 + x_2 = \frac{-2 a^2 m k}{b^2 + a^2 m^2}.$$

$$\therefore X = -\frac{a^2 m k}{b^2 + a^2 m^2}.$$

But  $(X, Y)$  satisfies  $y = mx + k$ .

$$\therefore Y = mX + k.$$

Eliminating  $k$  we have,

$$Y = \frac{-b^2}{a^2 m} X,$$

as the equation of the locus. This is the equation of a straight line through the center and hence is a diameter.

It follows immediately that if the slope of the chords is  $m$  and the slope of the diameter is  $m'$ ,  $m' = -\frac{1}{a^2 m}$

or  $mm' = -\frac{1}{a^2}$ .

Show for a hyperbola that  $mm' = \frac{1}{a^2}$ .

The proof just given assumes the parallel chords have a slope. It is left as an exercise to discuss the case in which they have no slope.

**Theorem 2.** *The locus of the middle points of a set of parallel chords of a parabola is a diameter.*

The proof of this theorem is left to the student.

If the equation of the parabola is  $y^2 = 4px$  the equation of the locus is  $y = \frac{2p}{m}x$  if the chords have a slope  $m$

and  $y = 0$ , if the chords have no slope.

**86. Conjugate diameters. Theorem.** *If one diameter bisects the chords parallel to a second, the second diameter bisects the chords parallel to the first.*

Diameters fulfilling the condition of this theorem are called conjugate diameters.

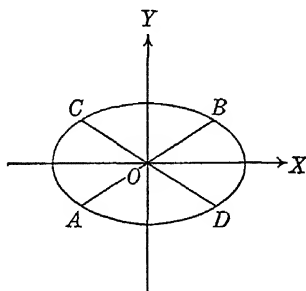


FIG. 106

To prove the theorem we proceed as follows:

Let the slope of  $AB$  be  $m$  and the slope of  $CD$  be  $m'$ .

By hypothesis the diameter of slope  $m'$ , *i.e.*,  $CD$ , bisects the chords of slope  $m$ , *i.e.*, chords parallel to  $AB$ . Therefore by Theorem 1,

$$m' = -\frac{b^2}{a^2m}, \text{ or } m = -\frac{b^2}{a^2m'}$$

This equation says that the diameter of slope  $m$ , *i.e.*,  $AB$ , bisects the chords of slope  $m'$ , *i.e.*, chords parallel to  $CD$ .

The proof just given assumes that both diameters have a slope. The discussion of the case in which one diameter has no slope is left as an exercise.

### Exercises

1. Prove that every pair of conjugate diameters of a circle are perpendicular to each other.
2. Why are there no conjugate diameters in the case of a parabola?
3. In the hyperbola  $\frac{x^2}{4} - \frac{y^2}{9} = 1$ , find the equations of the conjugate diameters if the slope of one diameter is 2.
4. In the ellipse  $\frac{x^2}{4} + \frac{y^2}{9} = 1$ , find the equations of the conjugate diameters if the slope of one diameter is 4.
5. Find the equations of the conjugate diameters of the hyperbola  $x^2 - 4y^2 = 10$ , one of which bisects the chord whose equation is  $3x - 2y = 4$ .

6. Find the equation of the diameter of the parabola  $y^2 = 8x$  which bisects all chords parallel to  $2x + 3y = 5$ .

7. In the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , if  $(x_1, y_1)$  are the coordinates of one extremity of a diameter, the coordinates of the other extremity are  $(-x_1, -y_1)$ ; prove that the coordinates of the extremities of the conjugate diameter are

$$\left(-\frac{ay_1}{b}, \frac{bx_1}{a}\right), \left(\frac{ay_1}{b}, -\frac{bx_1}{a}\right).$$

8. In the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , if  $(x_1, y_1)$  are the coordinates of one extremity of a diameter, the coordinates of the other extremity are  $(-x_1, -y_1)$ ; prove that the coordinates of the extremities of the conjugate diameter are

$$\left(\frac{ay_1}{b}, \frac{bx_1}{a}\right), \left(-\frac{ay_1}{b}, -\frac{bx_1}{a}\right).$$

9. Find the equation of a diameter of the ellipse  $x^2 + 4y^2 = 73$  if one end of its conjugate diameter is  $(3, 4)$ .
10. Find the equation of a chord of the ellipse  $4x^2 + 9y^2 = 36$  through the point  $(1, 2)$  which is bisected by the diameter  $y = x$ .
11. Find the length of the diameter of the hyperbola  $9x^2 - 4y^2 = 36$  which is conjugate to the diameter  $y = 2x$ .
12. Find the equation of the chord of the parabola  $y^2 = 10x$  which is bisected by the point  $(3, 2)$ .
13. Prove that the tangents at the extremities of a diameter of an ellipse are parallel to the conjugate diameter.
14. Prove that the sum of the squares of the lengths of any two

conjugate diameters of an ellipse is constant and equal to the sum of the squares of the axes.

15. Prove that two conjugate diameters of a hyperbola are never equal unless the hyperbola is rectangular and that in this case they are always equal.
16. Prove that the product of the focal radii of any point of an ellipse is equal to the square of half the diameter conjugate to the diameter through the point.
17. Prove that the tangent at the end of a diameter of a parabola is parallel to the system of chords which the diameter bisects.
18. Prove that the straight lines drawn from any point on an equilateral hyperbola to the extremities of any diameter make equal angles with the asymptotes.
19. If  $P_1$  and  $P_2$  are the extremities of a pair of conjugate diameters of an ellipse, prove that the normals at  $P_1$  and  $P_2$  and the perpendicular from the center to  $P_1P_2$  are concurrent.

**87. Harmonic division.** If the four points  $P_1, P_2, A_1, A_2$  are collinear and if the following equality is true,

$$\frac{P_1A_2}{A_2P_2} = \frac{P_1A_1}{A_1P_2} \qquad P_1 \qquad A_1$$

the four points are said to be *harmonic* and  $A_1, A_2$

FIG. 107

are said to divide the segment  $P_1P_2$  harmonically.

This proportion can be written,

$$\frac{A_2P_2}{P_2A_1} = \frac{A_2P_1}{P_1A_1}.$$

Hence, if two points  $A_1$ ,  $A_2$  divide the segment  $P_1P_2$  harmonically, then conversely, the two points  $P_1$ ,  $P_2$  divide the segment  $A_1A_2$  harmonically.

Let the coordinates of  $P_1$  and  $P_2$  be  $(x_1, y_1)$  and  $(x_2, y_2)$  respectively. Then the coordinates of  $A_1$  and  $A_2$  are

$$A_1: \quad x = \frac{r_2x_1 - r_1x_2}{r_2 - r_1}, \quad y = \frac{r_2y_1 - r_1y_2}{r_2 - r_1}$$

$$A_2: \quad x = \frac{r_2x_1 + r_1x_2}{r_2 + r_1}, \quad y = \frac{r_2y_1 + r_1y_2}{r_2 + r_1}$$

If we let the ratio  $\frac{r_1}{r_2} = R$ , we have

$$A_1: \quad x = \frac{x_1 - Rx_2}{1 - R}, \quad y = \frac{y_1 - Ry_2}{1 - R};$$

$$A_2: \quad x = \frac{x_1 + Rx_2}{1 + R}, \quad y = \frac{y_1 + Ry_2}{1 + R}$$

For any value of  $R$ , except  $R = \pm 1$ , these formulas determine two points which divide the segment  $P_1P_2$  harmonically.

### Exercises

1. Given  $P_1(2, 3)$  and  $P_2(6, 8)$ . Find two points both in the first quadrant which divide  $P_1P_2$  harmonically.
2. Given  $P_1(-2, 3)$  and  $P_2(4, 3)$ . Find two points, one in the first quadrant and one in the second which divide  $P_1P_2$  harmonically.

88. Poles and polars. Let a secant revolve about a

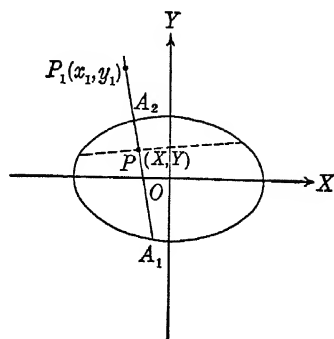


FIG. 108

point  $P_1(x_1, y_1)$  and cut the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  in points  $A_1$  and  $A_2$ . Let  $P(X, Y)$  be the point on the secant such that  $P_1P$  divides  $A_1A_2$  harmonically. The locus of  $P$  is called the polar of  $P_1$ .

From the last article the coordinates of

$$A_1 \text{ are } \left( \frac{x_1 + RX}{1 + R}, \frac{y_1 + RY}{1 + R} \right)$$

$$A_2 \text{ are } \left( \frac{x_1 - RX}{1 - R}, \frac{y_1 - RY}{1 - R} \right)$$

These points are on the ellipse and their coordinates satisfy  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . Substituting in the equation and simplifying, we have

$$b^2(x_1 + RX)^2 + a^2(y_1 + RY)^2 = a^2b^2(1 + R)^2,$$

$$b^2(x_1 - RX)^2 + a^2(y_1 - RY)^2 = a^2b^2(1 - R)^2.$$

Subtracting the equations eliminates the auxiliary variable  $R$  and we have

$$4 R b^2 x_1 X + 4 R a^2 y_1 Y = 4 R a^2 b^2,$$

which simplifies to

$$\frac{x_1 X}{a^2} + \frac{y_1 Y}{b^2} = 1,$$

which is the equation of a straight line, all or some part of which is the desired locus. If  $P_1$  is without the ellipse, the locus is the chord segment of this line. If  $P_1$  is within the ellipse, the locus is the entire line. If  $P_1$  is the center of the ellipse there is no locus.

It will be noted that the last equation is of the form of the tangent equation § 70. If  $P_1$  is on the curve the polar is the tangent.

Similarly we can show that the polar of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

for the point  $(x_1, y_1)$  is

$$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1,$$

and that the polar with respect to  $y^2 = 4 p x$  for the point  $(x_1, y_1)$  is

$$y_1 y = 2 p (x + x_1).$$

*Example 1.* Find the polar of  $(2, -1)$  with respect to  $x^2 + 4 y^2 = 20$ .

Solution: The polar of  $(x_1, y_1)$  is  $x_1 x + 4 y_1 y = 20$ .

$\therefore$  The polar of  $(2, -1)$  is  $2 x - 4 y = 20$

or  $x - 2 y = 10$ .

*Example 2.* Find the pole of  $2x - 3y = 10$  with respect to  $5x^2 + 7y^2 = 8$ .

Solution: The pole of  $(x_1, y_1)$  is  $5x_1x + 7y_1y = 8$ . Since this line must coincide with  $2x - 3y = 10$ , we have

$$\frac{5x_1}{2} = \frac{7y_1}{-3} = \frac{8}{10}$$

$$\text{Therefore } x_1 = \frac{8}{25}, \quad y_1 = -\frac{12}{35}.$$

$$\therefore \text{ The pole is } \left( \frac{8}{25}, -\frac{12}{35} \right).$$

### Exercises

1. Find the equation of the polar of:

a)  $(2, -5)$  with respect to  $4x^2 + 3y^2 = 6$ .

b)  $(2, -7)$  with respect to  $y^2 = 8x$ .

c)  $(-4, 1)$  with respect to  $x^2 + y^2 = 4$ .

2. Find the pole of:

a)  $3x - 2y = 7$  with respect to  $x^2 + 4y^2 = 7$ .

b)  $2x + 5y = 8$  with respect to  $y^2 = 8x$ .

c)  $3x - 7y = 10$  with respect to  $x^2 - 5y^2 = 6$ .

3. Prove that if two points are so situated that  $P_2$  lies on the polar of  $P_1$ , then  $P_1$  conversely, lies on the polar of  $P_2$ .

4. Prove that the polar of  $P_1$  with respect to an ellipse is parallel to the tangent at the point where the diameter through  $P_1$  cuts the ellipse.

5. Prove that the polar of any point  $P_1$  with respect to a parabola is parallel to the tangent at the point where a diameter through  $P_1$  cuts the parabola.
6. Prove that the line which joins any point to the center of a circle, is perpendicular to the polar of the point with respect to the circle.
7. Prove that the radius of a circle is a mean proportional between the distance from the center of the circle to any point and the distance from the center to the polar of that point.
8. Prove that the polar of a focus of an ellipse is the corresponding directrix.
9. Prove that the polar of the focus of a parabola is the directrix.
10. Prove that the polars of the same point with respect to two conjugate hyperbolas are parallel lines.
11. Prove that a focal chord of an ellipse is perpendicular to the line that joins its pole to the focus.

## CHAPTER X

### POLAR COORDINATES

**89. Polar coordinates.** We shall now introduce a new way of locating the position of a point in a plane. Let  $OA$  be a directed line in the plane, which we shall call the **initial line** or **polar axis**. It is customary to draw this line horizontally and direct it to the right. The point  $O$  is called the **pole** or the **origin**. Let any point in the plane be  $P$ ; draw line  $OP$ . The position of  $P$  is then fixed if we know  $\angle AOP = \theta$  and the distance  $OP = r$ . The two numbers  $r$  and  $\theta$  known as the **radius vector** and the **vectorial angle**, respectively, are called the **polar coordinates** of the point  $P$ .

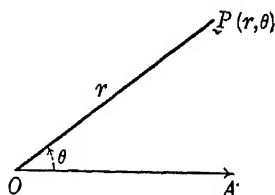


FIG. 109

The angle  $\theta$  is positive or negative according as to whether it is generated by a counter-clockwise or clockwise rotation. The positive direction on  $OP$  is the direction into which  $OA$  is rotated by a rotation through the angle  $\theta$ .

The following figures represent the points whose polar coordinates are  $(2, 45^\circ)$ ,  $(-2, 45^\circ)$ ,  $(2, -45^\circ)$ ,  $(-2, -45^\circ)$ .

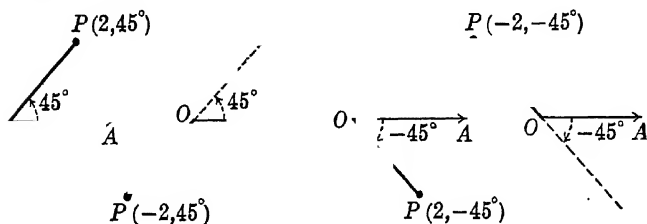


FIG. 110

A given point  $P$  has an unlimited number of sets of polar coordinates. If we confine ourselves to angles whose absolute value is less than  $360^\circ$ , a point will in general have four different sets of polar coordinates.

Thus  $(2, 45^\circ)$ ,  $(2, -315^\circ)$ ,  $(-2, -135^\circ)$ ,  $(-2, 225^\circ)$  are the same point.

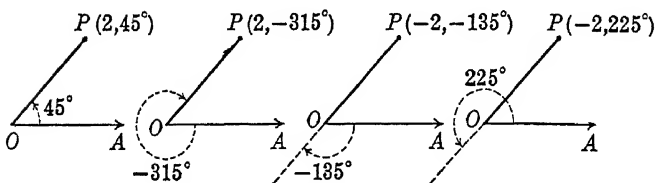


FIG. 111

### Exercises

- Plot the points whose polar coordinates are:
 

a) $(4, 60^\circ)$ .	b) $(2, 30^\circ)$ .	c) $(-2, 45^\circ)$ .
d) $(3, -150^\circ)$ .	e) $(-2, -100^\circ)$ .	f) $(-2, -180^\circ)$ .
g) $(-3, 120^\circ)$ .	h) $(-2, 270^\circ)$ .	i) $(3, 360^\circ)$ .
j) $(3, 240^\circ)$ .	k) $(-0, -400^\circ)$ .	l) $(5, 330^\circ)$ .
- For each of the points in Exercise 1, find all sets of polar coordinates for which  $|\theta| < 360^\circ$ .
- Where are all the points for which  $r = 4$ ?
- Where are all the points for which  $\theta = 30^\circ$ ?
- Show that the points  $(r, \theta)$  and  $(r, -\theta)$  are symmetric with respect to the polar axis.
- Show that the points  $(r, \theta)$  and  $(-r, \theta)$  are symmetric with respect to the pole.

7. Show that the points  $(r, \theta)$  and  $(r, \theta + 180^\circ)$  are symmetric with respect to the pole.
8. Use the law of cosines and find the distance between  $(3, 45^\circ)$  and  $(6, 105^\circ)$ .
9. Find the distance between  $(3, 30^\circ)$  and  $(10, 90^\circ)$ .
10. Prove that the distance between  $(r_1, \theta_1)$  and  $(r_2, \theta_2)$  is

$$\sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}.$$

**90. Graphs in polar coordinates.** To plot a graph in polar coordinates, we proceed by constructing a table of corresponding values of  $\theta$  and  $r$ . The point corresponding to each such pair of values is then plotted and a curve is drawn through these points.

*Example 1.* Plot the graph of  $r = \sin \theta$ . The following table of values is readily constructed.

$\theta$	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$135^\circ$	$150^\circ$	$180^\circ$	$210^\circ$	$225^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$315^\circ$	$330^\circ$	$360^\circ$
$r$	.00	.50	.71	.87	1.00	.87	.71	.50	.00	-.50	-.71	-.87	-1.00	-.87	-.71	-.50	.00

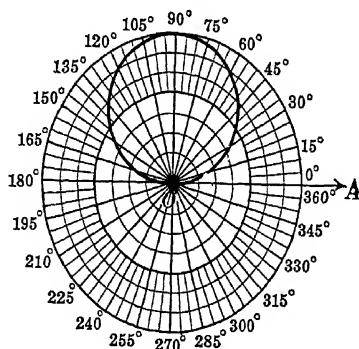


FIG. 112

In Fig. 112 the corresponding points are plotted and a smooth curve is drawn through them. It should be

noted that in this example each point is represented by two pairs of corresponding values; for example  $(.71, 45^\circ)$  and  $(-.71, 225^\circ)$  are the same point.

The curve appears to be a circle of diameter 1, and tangent to the polar axis at the origin.

### Exercises

Plot the graph of:

- |                             |                             |
|-----------------------------|-----------------------------|
| 1. $r = 2 \sin \theta.$     | 2. $r = 2 \cos \theta.$     |
| 3. $r^2 = \sin \theta.$     | 4. $r^2 = \cos \theta.$     |
| 5. $r = 2 \sin \theta + 5.$ | 6. $r = 2 \cos \theta + 5.$ |

**91. Locus of an equation.** The locus of an equation in polar coordinates is such that

1) Every point whose coordinates  $(r, \theta)$  satisfy the equation is on the locus or curve, and

2) A set of coordinates\* of every point on the locus or curve satisfies the equation.

As in rectangular coordinates, the amount of work is often greatly simplified if one employs symmetry. The following rules are obvious.

If a polar equation is left unchanged,

- a) when  $\theta$  is replaced by  $-\theta$ , the locus is symmetric with respect to the polar axis.
- b) when  $r$  is replaced by  $-r$ , the locus is symmetric with respect to the pole.

\* Not necessarily every set of coordinates. For example, the point  $(2, 60^\circ) = (-2, 240^\circ)$  is on the locus of  $r = 1 + 2 \cos \theta$ , but the second set of coordinates does not satisfy the equation.

- c) when  $\theta$  is replaced by  $180^\circ + \theta$ , the locus is symmetric with respect to the pole.
- d) when  $\theta$  is replaced by  $180^\circ - \theta$ , the locus is symmetric with respect to the line through the pole perpendicular to the polar axis.

One should always keep in mind that none of these rules is a *necessary* condition for symmetry.

*Example 1.* Discuss and plot the locus of the equation  $r = 4 \sin^2 \theta$ .

Solution: The locus is symmetric with respect to the pole, the polar axis and the line through the pole perpendicular to the polar axis.

If we plot points from  $\theta = 0^\circ$  to  $\theta = 90^\circ$  and employ symmetry, we have the locus of Fig. 113.

$\theta$	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$
$r$	0	1	2	3	4

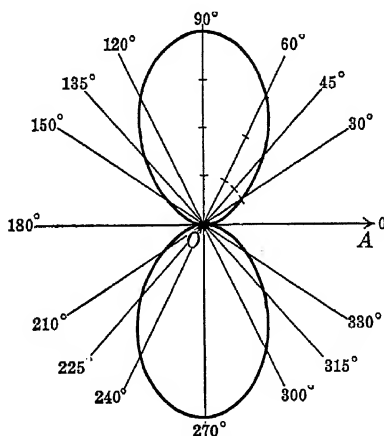


FIG. 113

It is advisable to check the branches constructed by symmetry, by substituting in the original equation the coordinates of at least one point on each branch, for by so doing, serious errors may be avoided.

*Example 2.* Discuss and plot the locus of the equation

$$r = 1 + 2 \cos \theta.$$

*Solution:* The curve is symmetric with respect to the polar axis. If we plot points from  $\theta = 0^\circ$  to  $\theta = 180^\circ$  and then employ symmetry, we obtain the complete graph.

$\theta$	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$135^\circ$	$150^\circ$	$180^\circ$
$r$	3	2.7	2.4	2	1	0	-.4	-.7	-1

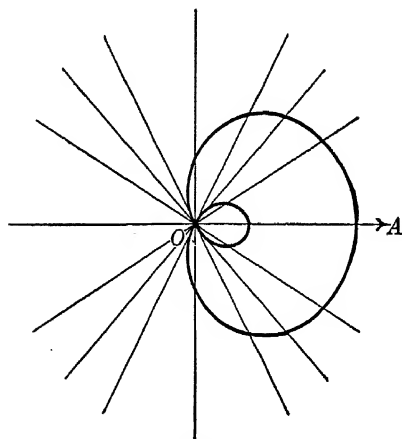


FIG. 114

## Examples

Plot the locus of each of the following equations, employing symmetry in each case.

1.  $r = 4.$

2.  $r = -4.$

3.  $r^2 = 16.$

4.  $\theta = 45^\circ.$

5.  $\theta = -45^\circ.$

6.  $r = 4 \cos \theta.$

7.  $r = -4 \cos \theta.$

8.  $r = 4 \sin \theta.$

9.  $r = -4 \sin \theta.$

10.  $r = 1 - \cos \theta.$

11.  $r = 1 + \cos \theta.$

12.  $r = 1 - \sin \theta.$

13.  $r = 1 + \sin \theta.$

14.  $r = 4 + \cos \theta.$

15.  $r = 4 - \cos \theta.$

16.  $r = 4 + \sin \theta.$

17.  $r = 4 - \sin \theta.$

18.  $r = 1 + 2 \sin \theta.$

19.  $r = 1 - 2 \sin \theta.$

20.  $r = 1 + 2 \cos \theta.$

21.  $r = 1 - 2 \cos \theta.$

22.  $r^2 = 9 \sin^2 \theta.$

23.  $r^2 = 9 \cos^2 \theta.$

24.  $r^2 = 25 \cos 2 \theta.$

25.  $r^2 = 25 \sin 2 \theta.$

26.  $r = 4 \sin 2 \theta.$

27.  $r = 4 \cos 2 \theta.$

28.  $r = 4 \sin 3 \theta.$

29.  $r = 4 \cos 3 \theta.$

30.  $r = \frac{4}{1 - \cos \theta}.$

31.  $r = \frac{4}{1 + \cos \theta}.$

32.  $r = \frac{4}{1 - \sin \theta}.$

33.  $r = \frac{4}{1 + \sin \theta}.$

34.  $r = 4 \tan \theta.$

35.  $r = 9 \sin^2 \frac{\theta}{2}.$

36.  $r = 4 \cot \theta.$

**92. Relation between rectangular and polar coordinates.** Take  $O$  the origin of a system of rectangular axes as the pole, and the positive half of the  $x$ -axis as the polar axis of a system of polar coordinates.

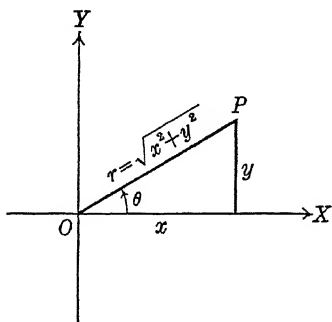


FIG. 115

Let  $(x, y)$  and  $(r, \theta)$  be respectively the rectangular and polar coordinates of any point  $P$ . Then  $x/r = \cos \theta$ ,  $y/r = \sin \theta$ .

Hence we have

$$(1) \quad \begin{aligned} x &= r \cos \theta, \\ y &= r \sin \theta. \end{aligned}$$

It is here assumed that the coordinates of  $P$  are so chosen that  $OP = r$  and angle  $XOP = \theta$ . This is always possible. If  $r$  is positive,  $x$  always has the sign of  $\cos \theta$  and  $y$  the sign of  $\sin \theta$ .

Conversely, if  $r$  is positive, we see from Fig. 115 that

$$(2) \quad \begin{aligned} r^2 &= x^2 + y^2, & \sin \theta &= \frac{y}{\sqrt{x^2 + y^2}} \\ \theta &= \arctan \left( \frac{y}{x} \right), & \cos \theta &= \frac{x}{\sqrt{x^2 + y^2}}. \end{aligned}$$

*Example 1.* Find the equation of  $x^2 - y^2 = 4$  in polar coordinates.

Solution: Replacing  $x$  and  $y$  by their values from (1), we have

$$r^2 \cos^2 \theta - r^2 \sin^2 \theta = 4$$

$$r^2 \cos 2\theta = 4.$$

*Example 2.* Find a set of polar coordinates for the point  $(-2, 2)$ .

Solution: From (2)  $r = \sqrt{4 + 4} = 2\sqrt{2}$ .

$$\theta = \arctan(-1).$$

$\therefore \theta = 135^\circ$  and a set of polar coordinates are

$$(2\sqrt{2}, 135^\circ).$$

### Exercises

Transform the following equations into equations in rectangular coordinates. In each case, state whether you think the locus is easier to sketch from the polar or from the rectangular equation.

1.  $r = 5$ .

2.  $r \sin \theta = 5$ .

3.  $r \cos \theta = 4$ .

4.  $r^2 \cos 2\theta = 4$ .

5.  $r = \theta$ .

6.  $r = 2a \sec \theta \tan \theta$ .

7.  $r = 4 \sin 2\theta$ .

8.  $r = \frac{1}{\theta}$ .

9.  $r = \frac{1}{1 - \cos \theta}$ .

10.  $r = 1 - \cos \theta$ .

Transform the following equations into equations in polar coordinates.

11.  $x^2 + y^2 = 4$ .

12.  $x + y = 10$ .

13.  $y^2 = 4x$ .

14.  $xy = 10$ .

15.  $9x^2 - 4y^2 = 36$ .

16.  $(y^2 + x^2 - 2x)^2 = x^2 + y^2$ .

17.  $x^3 + y^3 = 3axy$

18.  $x^3 = y^2(2 - x)$ .

19.  $y^3 = x^2$ .

20.  $x \cos \alpha + y \sin \alpha = p$ .

### 93. Standard equations in polar coordinates.

*The Straight Line.* Let  $BC$  be any straight line with  $ON = p$ , the perpendicular from  $O$  to the line, and  $\alpha$

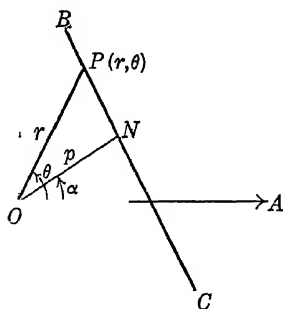


FIG. 116

the angle which the perpendicular makes with the polar axis. Let  $(r, \theta)$  be any point on the line. Then

$$\frac{ON}{OP} = \cos(\theta - \alpha) \text{ or } \cos(\alpha - \theta).$$

But  $\cos(\theta - \alpha) = \cos(\alpha - \theta)$ . Why? Therefore

$$(3) \quad r \cos(\theta - \alpha) = p$$

is the desired equation.

Conversely if (3) is true, by retracing our steps the point  $(r, \theta)$  is seen to lie on the given line. Therefore (3) is the equation of the desired locus.

There are two special cases which should be noted. If the line is perpendicular to the polar axis its equation is

$$r \cos \theta = p. \quad \text{Why?}$$

If the line is parallel to the polar axis its equation is

$$r \sin \theta = p. \quad \text{Why?}$$

*The circle.* Let the given circle have its center at

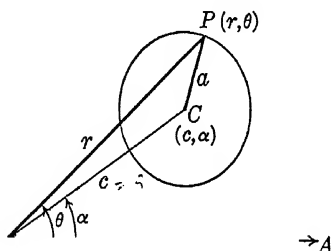


FIG. 117

$(c, \alpha)$ , and radius equal to  $a$ . If  $P(r, \theta)$  is any point on the circle, we have in  $\triangle OCP$ ,  $OC = c$ ,  $OP = r$ ,  $\angle COP = \pm(\theta - \alpha)$ , depending upon the position of the point  $P$ . Using the law of cosines and remembering that  $\cos(\theta - \alpha) = \cos(\alpha - \theta)$  we have

$$(4) \quad r^2 = c^2 + r^2 - 2cr \cos(\theta - \alpha),$$

as the equation of the circle.

If the circle passes through the pole ( $c = \pm a$ ), the equation becomes

$$(5) \quad r = \pm 2a \cos(\theta - \alpha).$$

If in addition the center  $C$  is on the polar axis ( $\alpha = 0^\circ$ ), the locus passes through the pole ( $c = \pm a$ ), and the equation becomes

$$(6) \quad r = \pm 2a \cos \theta.$$

If the center is at the pole, ( $c = 0$ ) the equation becomes

$$(7) \quad r = \pm a.$$

### Exercises

Sketch the following curves:

$$1. \ 4 = r \cos \left( \theta - \frac{\pi}{4} \right). \quad 2. \ 5 = r \cos \left( \theta - \frac{\pi}{3} \right).$$

$$3. \ 7 = r \cos \left( \frac{\pi}{2} - \theta \right). \quad 4. \ r = 5.$$

$$5. \ r = -5. \quad 6. \ r = 8 \cos \theta.$$

$$7. \ r = -8 \cos \theta. \quad 8. \ r = 8 \sin \theta.$$

$$9. \ r = -8 \sin \theta. \quad 10. \ r = 8 \cos \left( \theta - \frac{\pi}{3} \right).$$

$$11. \ r = 8 \cos \left( \theta - \frac{\pi}{4} \right). \quad 12. \ r = 8 (\sin \theta + \cos \theta).$$

**94. Polar equation of any conic.** Let  $DD'$  be the directrix,  $F$  the corresponding focus and  $e$  the eccentricity. Let the perpendicular through  $F$  to  $DD'$  meet it in  $M$ . Let the focus be the pole,  $MF = 2p$ , and  $P(r, \theta)$

any point on the curve. If the perpendicular from  $P$  to

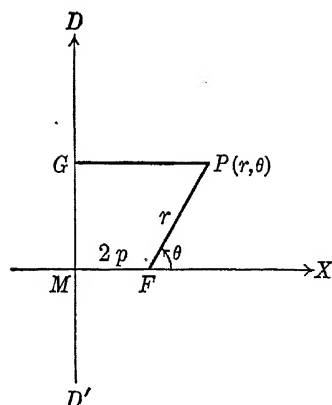


FIG. 118

$DD'$  meets it in  $G$ , we then have from the definition of a conic (§ 68)

$$PF = ePG,$$

$$r = e(2p + r \cos \theta),$$

$$(8) \text{ or } r = \frac{2ep}{1 - e \cos \theta},$$

which is the polar equation of a conic. If  $e < 1$ , it is the equation of an ellipse; if  $e = 1$ , a parabola; if  $e > 1$ , a hyperbola.

### Exercises

1. Derive the polar equation of the ellipse assuming the right-hand focus as the pole and the major axis as the polar axis.
2. Derive the polar equation of a hyperbola assuming the right-hand focus as the pole and the transverse axis as the polar axis.

**95. Special curves.** Certain locus problems are more readily solved by the use of polar coordinates, as the following examples will illustrate.

*The Limacon. The Cardioid.* Given a circle of diameter  $a$  and a fixed point  $O$  on it. On every line  $OM$

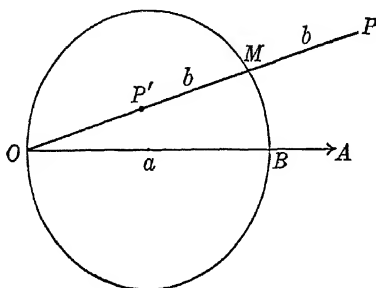


FIG. 119

through  $O$  meeting the circle again in  $M$  lay off constant distances  $MP = MP' = b$  in opposite directions from  $M$ . The locus of the points  $P$  and  $P'$  thus obtained is called the limacon of Pascal.

If the diameter  $OB (= a)$  through  $O$  is taken as the initial line and  $O$  as the pole, we have  $OM = a \cos \theta$  and the coordinates  $(r, \theta)$  of  $P$  are determined by

$$(9) \quad r = a \cos \theta + b.$$

If we change  $\theta$  to  $\theta + \pi$ ,  $r$  becomes  $-a \cos \theta + b$ , which gives the point  $P'$ . Hence, (9) is the equation of the locus of both  $P$  and  $P'$ . If  $\theta$  is increased by  $2\pi$ , we get the same point as when we use  $\theta$  itself; the whole curve is obtained by letting  $\theta$  vary from  $0$  to  $2\pi$ . If  $\theta$  be changed to  $-\theta$ , we get the same value of  $r$ ; the curve is, therefore, symmetric with respect to the initial line.

If  $b > a$ ,  $r$  is always positive, and decreases from

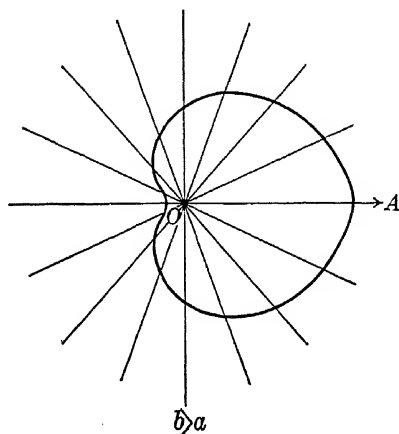


FIG. 120

$b + a$  to  $b - a$  as  $\theta$  increases from 0 to  $\pi$ . (Fig. 120.)

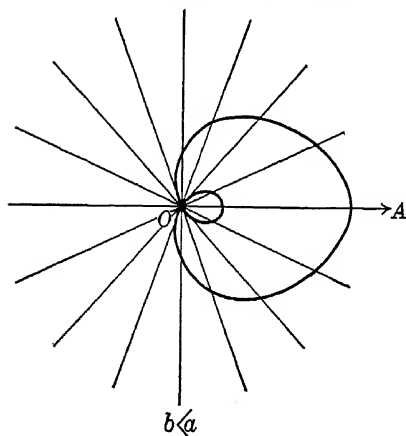


FIG. 121

If  $b < a$ ,  $r$  is 0 when  $\cos \theta = -\frac{b}{a}$ . If  $\theta$  increases

from 0 to  $2\pi$ ,  $r$  decreases, becoming 0 and then negative, reaching its smallest algebraic value  $b - a = -(a - b)$  when  $\theta = \pi$ . The curve then crosses itself at  $O$ . See Fig. 121.

Finally, when  $b = a$ ,  $r$  is 0 when  $\theta = \pi$ , but is never negative. In this case the equation is

$$(10) \quad r = a(1 + \cos \theta).$$

This special case of the limaçon is called the **cardioid**.

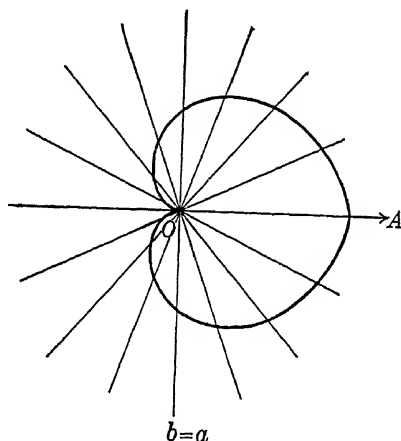


FIG. 122

The **spiral of Archimedes** is the locus of a point such that its radius vector is proportional to its vectorial angle. Therefore its equation is

$$(11) \quad r = k\theta,$$

where  $k$  is a constant.\*

\* In this example, and in those that follow, it is usual to express the angle  $\theta$  in radians; but this is not necessary, since the same result can be obtained by choosing a different value for  $k$  if  $\theta$  is expressed in degrees.

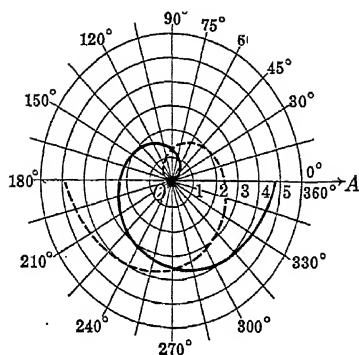


FIG. 123

The form of the equation shows that the locus passes through the pole, and that the radius vector increases without limit as the number of revolutions increases without limit. Fig. 123 represents a portion of the locus for  $k = \frac{1}{75}$ , with  $\theta$  expressed in degrees.

The **hyperbolic or reciprocal spiral** is the locus of a point such that its radius vector is inversely proportional to its vectorial angle. The equation of the locus is therefore

$$(12) \quad r = \frac{k}{\theta},$$

where  $k$  is a constant. Fig. 124 represents a portion of

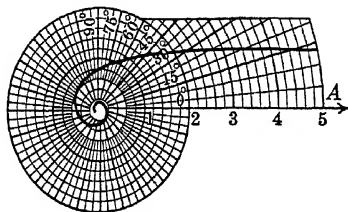


FIG. 124

the graph for  $k = 70$  and for positive values of  $\theta$  expressed in degrees.

The **logarithmic spiral** is the locus of a point such that the logarithm of its radius vector is proportional to its vectorial angle, *i.e.*,

$$(13) \quad \log r = k \theta,$$

where  $k$  is a constant. If the base of the system of

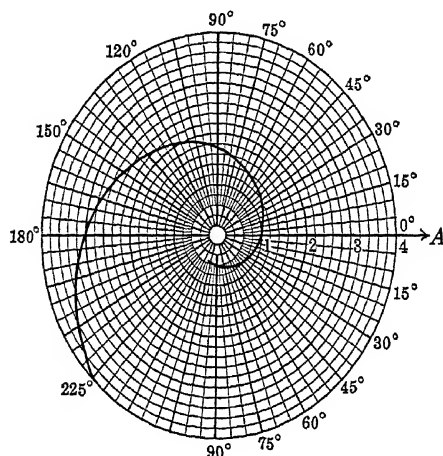


FIG. 125

logarithms is  $b$ , the equation may be written in the form  $r = b^{k\theta}$ . Fig. 125 represents a portion of this locus when  $b = 3$ , for  $k = \frac{1}{180}$ , with  $\theta$  expressed in degrees.

### Exercises

1. Draw the **parabolic spiral** which is defined by the equation  $r^2 = k \theta$ . Take  $k = .1$  with  $\theta$  in degrees and use only positive values of  $r$ .

2. Draw the **lituus** which is defined by the equation  $r^2 = \frac{k}{\theta}$ .  
Take  $k = 90$  with  $\theta$  in degrees, and use only positive values of  $r$ .
3. Find by polar coordinates the locus of the center of a circle which passes through a fixed point  $O$  and has a radius of 4.
4. Find by polar coordinates the locus of the middle points of all chords drawn through a fixed point on a circle.
5.  $OA$  is a fixed diameter of a fixed circle. At  $A$  a tangent is drawn, while about  $O$  a secant revolves which meets the tangent in  $B$  and the circle in  $C$ . Find the locus of the point  $P$  so situated on the segment  $OB$  that  $OP = CB$ .
6. Through a fixed point  $O$  on a fixed circle a variable secant  $OP$  is drawn, cutting the circle in  $A$ . If  $AP = 3 OA$ , find the locus of  $P$ .

### MISCELLANEOUS EXERCISES

1. Show that the polar equation of the curve whose equation is  $Ax^2 + By^2 + C = 0$  can be written in the form

$$r^2 = \frac{-C}{A \cos^2 \theta + B \sin^2 \theta}$$

2. Show that the polar equation of the curve whose equation is  $x^2 - y^2 = a^2$  can be written  $r^2 \cos 2\theta = a^2$ .
3. Show that the cartesian equation of the **lemniscate**  $r^2 = a^2 \cos 2\theta$ , can be written  $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$ .
4. Transform to cartesian coordinates the equation  $r \sin^2 \frac{\theta}{2} = a$ . Determine the nature of the curve.

5. Show that the equation  $\frac{x}{a} + \frac{y}{b} = 1$  can be written

$$r = \frac{ab}{a \sin \theta + b \cos \theta}$$

6. Derive a formula for the area of a triangle  $OP_1P_2$  in polar coordinates,  $O$  being the pole.
7. A comet moves in a parabolic orbit with the sun as focus. When the comet is 40,000,000 miles from the sun, the line from the sun to the comet makes an angle of  $\pi/3$  with the axis. How near does the comet come to the sun?
8. An ellipse, which has a focus at the pole and its major axis along the initial line, passes through the points  $(3, \pi/3)$  and  $(2, \pi/2)$ . What is its equation? Where is the second focus?
9. The cissoid of Diocles.  $OB$  is a fixed diameter of a circle and a variable line through  $O$  meets the circle in  $M$  and the tangent at  $B$  in  $N$ . Find in polar coordinates the equation of the locus of  $P$  on  $ON$ , such that  $PN = OM$ .

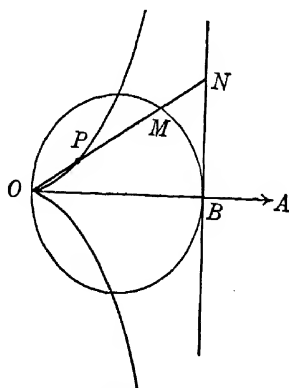


FIG. 126

10. **The conchoid of Nicomedes.** Given a fixed point  $O$  (the pole) and a fixed line  $d$  (the base) which does not pass through  $O$ . A variable line through  $O$  meets  $d$  in a point  $M$  and from  $M$  a constant distance  $b$  is laid off on  $OM$  in both directions. Find in polar coordinates the equation of the locus of the two points thus obtained.

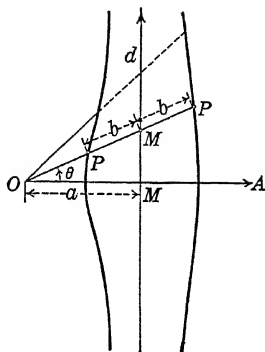


FIG. 127

## CHAPTER XI

### PARAMETRIC EQUATIONS

**96. Parametric equations.** We have seen in the discussion of locus problems, that as a point  $P(x, y)$  moves along a given curve, not only do the  $x$ - and  $y$ -coordinates of the point vary, but so do many other quantities connected with this point, as for example the angle  $\theta$  which  $OP$  makes with the  $x$ -axis. These other variables are called in § 84 **auxiliary variables** or **parameters**. If the coordinates  $x$  and  $y$  of a point are expressed in terms of a parameter, the equations are called **parametric equations**.

Thus, if  $P(x, y)$  is any point on the circle whose center

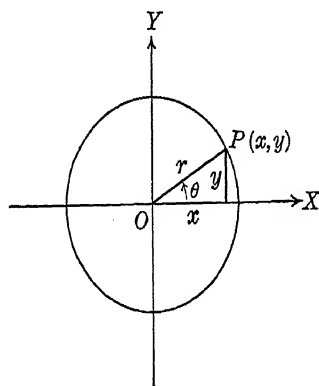


FIG. 128

is at the origin  $O$  and whose radius is  $r$ , then

$$(1) \quad \begin{aligned} x &= r \cos \theta, \\ y &= r \sin \theta, \end{aligned}$$

where  $\theta$  is the angle  $XOP$ ; as  $\theta$  varies from  $0^\circ$  to  $360^\circ$ , the point  $P$  traces the circle.

If  $\theta$  is eliminated between these equations (square and add them) the resulting equation which we obtain, namely,  $x^2 + y^2 = r^2$ , is the rectangular equation of the circle.

It is important to notice that any given curve may have as many sets of parametric equations as we please. For example

$$\begin{cases} x = t \\ y = 4t, \end{cases} \quad \begin{cases} x = \frac{t}{2} \\ y = 2t, \end{cases} \quad \begin{cases} x = \frac{t}{4} \\ y = t, \end{cases} \quad \begin{cases} x = \frac{t}{35} \\ y = \frac{4t}{35}, \end{cases}$$

are parametric equations of the line whose rectangular equation is  $y = 4x$ .

**97. Parametric equations of an ellipse.** Draw two concentric circles having  $A_2A_1 = 2a$  and  $B_2B_1 = 2b$  as diameters. Let  $ORS$  be any half-line issuing from

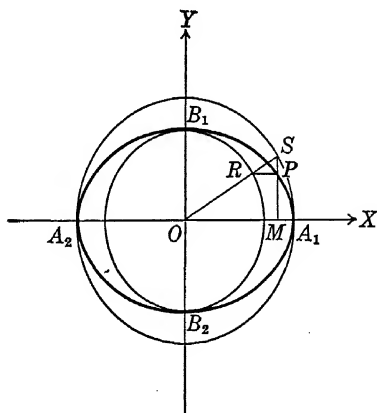


FIG. 129

the center  $O$  and meeting the circles in  $R$  and  $S$ . A line through  $R$  parallel to  $A_2A_1$  and a line through  $S$  parallel to  $B_1B_2$  meet in a point  $P$ . The locus of  $P$  we shall presently show is an ellipse. If angle  $XOS = \theta$ , the coordinates  $OM = x$  and  $MP = y$  of  $P$ , referred to  $OX$  and  $OY$  as axes, satisfy the relations,

$$(2) \quad \begin{aligned} x &= a \cos \theta, \\ y &= b \sin \theta. \end{aligned}$$

If  $\theta$  is eliminated between these equations we obtain  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . Therefore equations (2) constitute a pair of parametric equations of the ellipse. To every value of  $\theta$  corresponds one point of the ellipse, the coordinates for which are given by (2). As the parameter  $\theta$  varies from  $0^\circ$  to  $360^\circ$ , the point  $(x, y)$  traces out the entire curve. The angle  $XOS = \theta$  is called the **eccentric angle** of the point  $P$ .

By drawing various lines  $ORS$ , any number of points on the ellipse can be constructed.

### Exercises

1. Show that  $x = 2t$ ,  $y = 3 - 6t$  are parametric equations of a straight line.
2. Show that  $x = pt^2$ ,  $y = 2pt$  are parametric equations of the parabola  $y^2 = 4px$ .
3. Show that  $x = \frac{2az}{1+z^2}$ ,  $y = \frac{a(1-z^2)}{1+z^2}$  are parametric equations of a circle.

4. Show that  $x = a \sec \theta$ ,  $y = b \tan \theta$  are parametric equations of a hyperbola.
5. Find a pair of parametric equations for the rectangular hyperbola  $x^2 - y^2 = a^2$ .
6. Prove that  $x = A \cos \theta + B \sin \theta$ ,  $y = A \sin \theta - B \cos \theta$  are parametric equations of a circle.
7. Prove that  $x = \frac{3at}{1+t^3}$ ,  $y = \frac{3at^2}{1+t^3}$  are parametric equations of the curve  $x^3 + y^3 - 3axy = 0$ .
8. Prove that  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta$  are parametric equations of the curve  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ .
9. Find the equation of the tangent to
  - (a)  $b^2x^2 + a^2y^2 = a^2b^2$ , at  $x_1 = a \cos \theta_1$ ,  $y_1 = b \sin \theta_1$ ;
  - (b)  $b^2x^2 - a^2y^2 = a^2b^2$ , at  $x_1 = a \cos \theta_1$ ,  $y_1 = b \sin \theta_1$ ;
  - (c)  $y^2 = 4px$ , at  $x_1 = pt_1^2$ ,  $y_1 = 2pt_1$ .
10. Find the point of intersection of the tangents to  $y^2 = 4px$  at  $(pt_1^2, 2pt_1)$  and  $(pt_2^2, 2pt_2)$ .

**98. Graph of parametric equations.** If one assigns a series of values to the parameter and determines the series of corresponding pairs of values of  $x$  and  $y$ , these values can be interpreted as the coordinates of points on the curve. If these points are plotted and a curve is drawn through them, we have the graph of the curve whose parametric equations were given.

*Example.* Draw the graph of the curve, a pair of whose parametric equations are  $x = t$ ,  $y = t^2 - 1$ .

Solution: Assigning values to  $t$  and computing the corresponding values of  $x$  and  $y$ , we form the table

$t$	-4	-3	-2	-1	0	1	2	3	4
$x$	-4	-3	-2	-1	0	1	2	3	4
$y$	15	8	3	0	-1	0	3	8	15

Plotting these points we have the graph in Fig. 130.

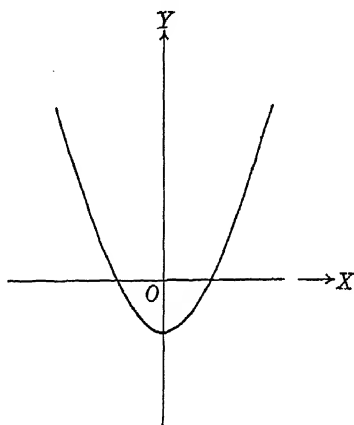


FIG. 130

**99. Time as a parameter.** If a point moves in a plane, the point occupies a certain position  $(x, y)$ , at every instant of time  $t$ . This fact is denoted by saying that  $x$  and  $y$  are functions of the time  $t$ .

Equations of this type arise frequently in mechanics where one wishes to describe the motion of a body which is subjected to various forces.

For example: Suppose a body is projected from the point  $O$  (origin) in a vertical plane, at time  $t = 0$ , with an initial velocity  $v_0$  and making an angle  $\alpha$  with the

horizontal ( $x$ -axis). Then its position at the end of  $t$  seconds, the resistance of the air being neglected, may

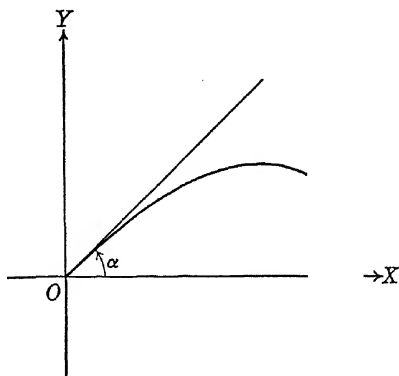


FIG. 131

be determined as follows. The  $x$  or horizontal component of the velocity is  $v_0 \cos \alpha$ , and the  $y$  or vertical component is  $v_0 \sin \alpha$ . If the resistance of the air is neglected the distance traversed is the velocity multiplied by the time, and hence we have, recalling the fact that the distance passed over by a falling body is  $\frac{1}{2}gt^2$ ,

$$x = (v_0 \cos \alpha) t,$$

$$y = (v_0 \sin \alpha) t - \frac{1}{2}gt^2.$$

### Exercises

Sketch the following curves from their parametric equations:

- |                   |                    |                 |
|-------------------|--------------------|-----------------|
| 1. $x = 3t,$      | 2. $x = 3t + 1,$   | 3. $x = t + 1,$ |
| $y = 7t.$         | $y = 2t - 5.$      | $y = t^2.$      |
| 4. $x = t^2 + 1,$ | 5. $x = t,$        | 6. $x = t^2,$   |
| $y = t^2 - 1.$    | $y = \frac{1}{t}.$ | $y = t^3.$      |

7.  $x = 3 \sin \theta$ ,  $y = 3 \cos \theta$ .  
 8.  $x = 4 \tan \theta$ ,  $y = 4 \sec \theta$ .  
 9.  $x = 4t$ ,  $y = 20 - 16t^2$ .  
 10.  $x = t^2 + 2$ ,  $y = t^3 - 1$ .

11. Prove that the equation of the path of the projectile in § 99 written in rectangular coordinates is

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2} \sec^2 \alpha.$$

12. Prove that the range of the projectile in Ex. 11 is

$$R = \frac{v_0^2}{g} \sin 2\alpha.$$

13. Find the locus of a point  $P$  on a circle which rolls along a fixed line. Outline of proof follows:

Take for the origin the point  $O$  where the moving point  $P$  touches the fixed line. If  $r$  is the radius of the circle and the angle  $PCD$  (Fig. 132) is  $\theta$  radians, then  $PD = r \sin \theta$ ,  $DC = r \cos \theta$  and  $OB = \text{arc } BP = r\theta$ .

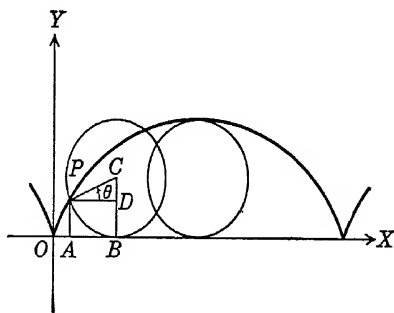


FIG. 132

Now if  $P$  is denoted by the coordinates  $(x, y)$ ,

$$x = OA = OB - AB = OB - PD = r\theta - r\sin\theta = r(\theta - \sin\theta),$$

$$y = AP = BC - DC = r - r\cos\theta = r(1 - \cos\theta).$$

Therefore

$$x = r(\theta - \sin\theta),$$

$$y = r(1 - \cos\theta),$$

are parametric equations of the curve traced by the point  $P$ . This curve is known as the *cycloid*.

14. Find the locus of a point  $P$  on a circle of radius  $a$  which rolls on the inside of a circle of radius  $4a$ . Outline of proof:

Take the center of the fixed circle as the origin and let the  $x$ -axis pass through a point  $M$  where the moving point  $P$  touches the large circle.

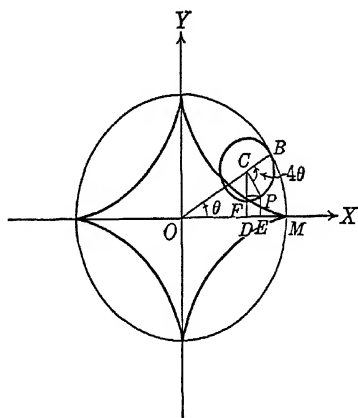


FIG. 133

Let angle  $MOB = \theta$  radians. Now we have arc  $PB =$  arc  $MB = 4a\theta$  and arc  $PB = a \times$  angle  $PCB$ .

Therefore

$$a \times \text{angle } PCB = 4 a \theta,$$

or

$$\text{angle } PCB = 4 \theta.$$

But

$$\angle OCD + \angle DCP + \angle PCB = \pi.$$

Therefore

$$\frac{\pi}{2} - \theta + \angle DCP + 4 \theta = \pi,$$

i.e.,

$$\angle DCP = \frac{\pi}{2} - 3 \theta.$$

Now if the point  $P$  is denoted by  $(x, y)$  we have

$$x = OE = OD + DE = OD + FP = OC \cos \theta + CP \sin \left( \frac{\pi}{2} - 3 \theta \right)$$

$$= 3 a \cos \theta + a \cos 3 \theta = 4 a \cos^3 \theta, *$$

$$y = EP = DC - FC = OC \sin \theta - CP \cos \left( \frac{\pi}{2} - 3 \theta \right)$$

$$= 3 a \sin \theta - a \sin 3 \theta = 4 a \sin^3 \theta; *$$

That is

$$x = 4 a \cos^3 \theta, \quad y = 4 a \sin^3 \theta.$$

This curve is called the *four-cusped hypocycloid*.

15. A circle of radius  $r$  rolls on the inside of a circle of radius  $a$   
Find the locus of a point  $P$  on the moving circle.

\* Prove that  $\cos 3 \theta = \cos (2 \theta + \theta) = 4 \cos^3 \theta - 3 \cos \theta,$   
 $\sin 3 \theta = \sin (2 \theta + \theta) = 3 \sin \theta + 4 \sin^3 \theta.$

*Ans.* The hypocycloid

$$x = (a - r) \cos \theta + r \cos \frac{a - r}{r} \theta,$$

$$y = (a - r) \sin \theta - r \sin \frac{a - r}{r} \theta,$$

where  $\theta$  is the same as in Ex. 14.

16. A circle of radius  $r$  rolls on the outside of a circle of radius  $a$ . Find the locus of a point  $P$  on the moving circle.

*Ans.* The epicycloid:

$$x = (a + r) \cos \theta + r \cos \frac{a + r}{r} \theta,$$

$$y = (a + r) \sin \theta - r \sin \frac{a + r}{r} \theta,$$

where  $\theta$  is the same as in Ex. 14.

## CHAPTER XII

### SOLID ANALYTIC GEOMETRY

**100. Cartesian coordinates.** Most of the problems of analytic geometry in the plane have counterparts in the geometry of three dimensions. At the same time, geometry of three dimensions, when extensively studied, is found to have many new complications.

Cartesian coordinates in three-dimensional space are a ready generalization of cartesian coordinates in the plane. We assume three straight lines called  $x$ ,  $y$  and  $z$  axes, intersecting at a point. On each of these lines there is a number scale, on each of which the zero point is the point common to the three lines. In this course we shall always assume the axes rectangular, that is, each one is perpendicular to each of the others, and positive directions will be chosen as indicated in the figure below.

The axes drawn as we have drawn them, and taken in the cyclic order  $x y z$ , form what is known as a left-handed system.

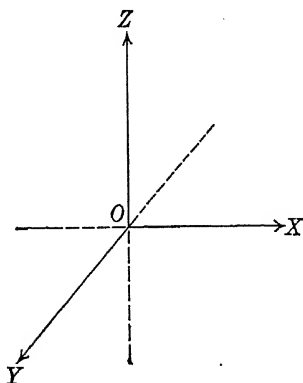


FIG. 134

The axes taken in pairs determine three planes known as coordinate planes, namely, the  $xy$ -plane, the  $xz$ -plane and the  $yz$ -plane. These planes divide the space into eight parts called octants and frequently numbered as follows. Mark as number I the octant whose bounding edges are the positive directions on the  $x$ ,  $y$  and  $z$  axes. Then proceed in a counter-clockwise sense, numbering the other three octants above the  $xy$ -plane II, III, IV; number V the octant directly below I and proceed in a counter-clockwise direction until all are numbered. Points lying in one of the coordinate planes will not be thought of as in any octant. Let  $P$  be any point in

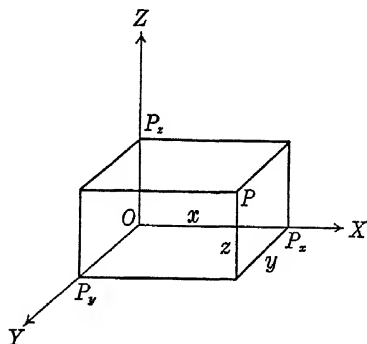


FIG. 135

space. Through  $P$  pass planes perpendicular to the  $x$ ,  $y$  and  $z$  axes, meeting the axes in the points  $P_x$ ,  $P_y$  and  $P_z$  respectively.

Then the coordinates of  $P$  are the algebraic values of the directed line segments  $OP_x$ ,  $OP_y$  and  $OP_z$ . We call these respectively the  $x$ ,  $y$  and  $z$  coordinates and shall always give them in the order named. We represent the point by  $(x, y, z)$ . Now every point in space has a

unique set of coordinates, and conversely every triple of numbers  $(x, y, z)$  determines one and only one point of which they are the coordinates. The student should make this quite clear to himself.

### Exercises

1. In what octant does a point lie, if all its coordinates are positive? if  $x$  is positive,  $y$  negative and  $z$  positive? if all the coordinates are negative?
2. In each of the following cases draw a figure showing the position of the point. The figure must be constructed so as to give the appearance of three dimensions.  $(3, 4, 7)$ ,  $(-2, 4, -2)$ ,  $(7, 9, -3)$ ,  $(3, 6, 1)$ ,  $(-1, -2, -3)$ ,  $(-8, 4, 2)$ ,  $(0, 0, 2)$ ,  $(2, 0, 0)$ ,  $(4, -2, 6)$ .
3. What are the relative positions of the following eight points  $(a, b, c)$ ,  $(a, b, -c)$ ,  $(a, -b, c)$ ,  $(-a, b, c)$ ,  $(a, -b, -c)$ ,  $(-a, b, -c)$ ,  $(-a, -b, c)$ ,  $(-a, -b, -c)$ ?

**101. Distance between two points.** The distance between two points  $P_1$  and  $P_2$  in terms of their coordi-

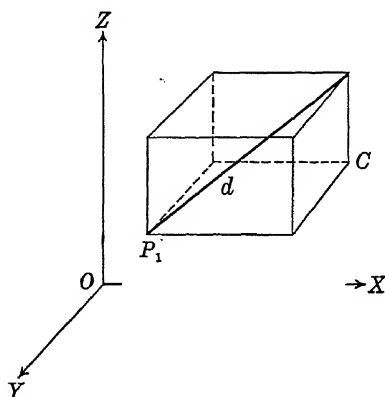


FIG. 136

nates is found by the Pythagorean theorem as in plane analytic geometry. The distance  $d$  is the length  $P_1P_2$  of the diagonal of the rectangular parallelopiped which has  $P_1B = |x_2 - x_1|$ ,  $BC = |y_2 - y_1|$  and  $CP_2 = |z_2 - z_1|$  respectively for lengths of its sides. Now

$$d^2 = \overline{P_1P_2}^2 = \overline{P_1B}^2 + \overline{BC}^2 + \overline{CP_2}^2,$$

$$(1) \quad \therefore d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

**102. Point of division.** The problem of finding the coordinates of a point which divides the line segment joining two given points in a given ratio is solved by the same method and leads to the formulas of the same form

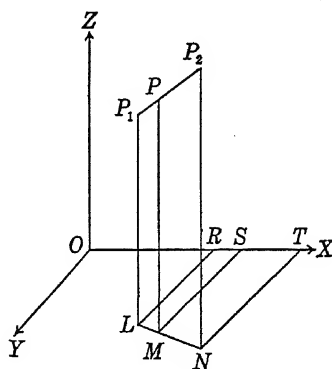


FIG. 137

as in the corresponding problem in two dimensions. It is left as an exercise for the student to prove

$$x = \frac{r_2x_1 + r_1x_2}{r_1 + r_2},$$

$$(2) \quad y = \frac{r_2 y_1 + r_1 y_2}{r_1 + r_2},$$

$$z = \frac{r_2 z_1 + r_1 z_2}{r_1 + r_2}$$

**103. Direction cosines.** Consider any directed line  $l$  in space and draw a line  $l_2$  through the origin parallel to it. Let the directions of  $l$  and  $l_2$  be the same. The positive angles made by the positive direction of  $l_2$  respectively with the positive directions on each of the axes are called the direction angles of the directed line  $l$ . Denote the angles made by  $l_2$  with the  $X$ ,  $Y$  and  $Z$  axes in order by  $\alpha$ ,  $\beta$ ,  $\gamma$ . Then  $\cos \alpha = \lambda$ ,  $\cos \beta = \mu$ ,  $\cos \gamma = \nu$  are called the direction cosines of the directed line  $l$ .

In much of solid analytic geometry lines are not directed. However, we shall speak of direction angles and direction cosines of the line. The angles  $\alpha$ ,  $\beta$ ,  $\gamma$  can be replaced by  $180^\circ - \alpha$ ,  $180^\circ - \beta$  and  $180^\circ - \gamma$  respectively and then  $\lambda$ ,  $\mu$ ,  $\nu$ , must be replaced by  $-\lambda$ ,  $-\mu$ ,  $-\nu$  respectively. It is frequently quite unnecessary to specify which of these sets of direction angles is chosen, but we must remember to be consistent in our usage in a given problem. Convenient formulas for determining direction cosines of a line are the following. Let  $P_1$  and  $P_2$  be two points on the line and let  $d$  be the length of the line segment  $P_1P_2$ . Then:

$$\lambda = \cos \alpha = \frac{x_2 - x_1}{d},$$

$$(3) \quad \mu = \cos \beta = \frac{y_2 - y_1}{d},$$

$$\nu = \cos \gamma = \frac{z_2 - z_1}{d}$$

Two important theorems concerning direction cosines are as follows.

**Theorem 1.** *The sum of the squares of a set of direction cosines for any line is unity.*

From formulas (3)

$$\lambda^2 + \mu^2 + \nu^2 = \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}{d^2}.$$

Substitute for  $d^2$  from (1) and we have

$$(4) \quad \lambda^2 + \mu^2 + \nu^2 = 1.$$

**Theorem 2.** Any three numbers  $a, b, c$  (not all zero) are proportional to the direction cosines of some line.

For,  $P(a, b, c)$  is a point, and the direction cosines of the directed line  $OP$  are

$$\begin{aligned} \lambda = \cos \alpha &= \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \\ (5) \quad \mu = \cos \beta &= \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \\ \nu = \cos \gamma &= \frac{c}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned}$$

The direction cosines of  $OP$  are evidently proportional to  $a, b, c$ , and they may be found by dividing  $a, b$ , and  $c$ , respectively, by  $\sqrt{a^2 + b^2 + c^2}$ .

### Exercises

Find the distance between the pair of points and the direction cosines of the line joining them in each of the following examples.

1.  $(1, 2, 3), (4, -1, 6)$ .
2.  $(7, -1, -2), (-1, -4, 3)$ .

3.  $(-1, -1, -1)$ ,  $(1, 1, 1)$ .
4.  $(0, 0, 0)$ ,  $(6, 3, 2)$ .
5.  $(0, 0, -10)$ ,  $(10, 0, 0)$
6. What are the coordinates of the point which divides the directed line segment from  $(1, 2, -1)$  to  $(3, -1, 4)$  into two portions whose algebraic values are in the ratio 2 to 3?  $-1$  to 2?
7. If  $\lambda : \mu : \nu = 3 : 2 : 1$  where  $\lambda, \mu, \nu$  are the direction cosines of a line, what are the values of  $\lambda, \mu, \nu$ ?
8. Find the direction cosines of a line which is equally inclined to all the axes.
9. Find the equation of the locus of points equidistant from  $(-1, 2, 3)$  and  $(4, 6, 8)$ .
10. Describe the locus of the equation

$$x^2 + y^2 + z^2 = 25.$$

**104. Orthogonal projection.** It is advisable at this point to review the notion of orthogonal projection, inasmuch as it is very important in establishing some of our formulas.

A point,  $P$ , is orthogonally projected onto a line  $l$  if a perpendicular from the point is drawn to the line. The foot of this perpendicular,  $\bar{P}$ , is called the orthogonal projection of  $P$  on  $l$ .

The projection of a directed line segment  $P_1P_2$  on a line is the directed segment  $\bar{P}_1\bar{P}_2$  on  $l$  where  $\bar{P}_1$  and  $\bar{P}_2$  are the projections on  $l$  of  $P_1$  and  $P_2$  respectively. The points  $\bar{P}_1$  and  $\bar{P}_2$  are frequently most easily located by passing planes through  $P_1$  and  $P_2$  respectively, each perpendicular to  $l$ .

It is immediate from trigonometry that  $\overline{P_1P_2} = P_1P_2 \cos \theta$  where  $\theta$  is an angle between  $P_1P_2$  and  $l$ .

**Theorem:** Given a broken line  $ABCD \dots N$  in space. The sum of the projections on a line of the directed straight line segments which constitute the broken line, is algebraically equal to the projection of the straight line segment  $AN$  on  $l$ .

The proof is a consequence of the definition of addition of directed line segments lying on a given line. The accompanying figure is self-explanatory and quite typical of any such situation.

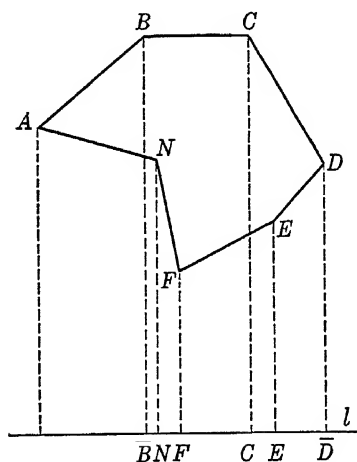


FIG. 138

Here

$$\overline{AB} + \overline{BC} + \overline{CD} + \overline{DE} + \overline{EF} + \overline{FN} = \overline{AN}.$$

As a corollary to this theorem we have that, if the broken line  $ABC \dots N$  forms a closed polygon, its projection on any line is zero.

**105. Angle between two lines.** The angle between two directed lines that do not meet, is defined to be the angle between two directed lines drawn through any point parallel to the given lines and having the same directions. There is ambiguity as to what angle is meant, inasmuch as if  $\theta$  is one possibility,  $-\theta$  is another,  $2\pi + \theta$  is still another and so on. However, all these angles have the same cosine and consequently there is no uncertainty in the following formula.

**Theorem:** *If the direction cosines of two directed lines  $l_1$  and  $l_2$  are  $\lambda_1, \mu_1, \nu_1$ , and  $\lambda_2, \mu_2, \nu_2$  respectively, then the cosine of the angle  $\theta$  between  $l_1$  and  $l_2$  is given by the formula*

$$(6) \quad \cos \theta = \lambda_1 \lambda_2 + \mu_1 \mu_2 + \nu_1 \nu_2.$$

Proof: Let  $\alpha_1, \beta_1, \gamma_1$  and  $\alpha_2, \beta_2, \gamma_2$  be the direction angles of two directed lines  $l_1$  and  $l_2$  and  $\theta$  the angle between them.

Draw lines  $l'_1$  and  $l'_2$  through the origin and parallel to the given lines. The angle between  $l'_1$  and  $l'_2$  is  $\theta$ .

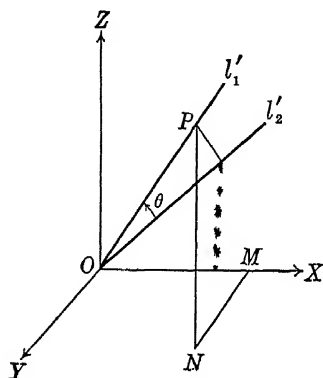


FIG. 139

Let  $P(x, y, z)$  be any point on  $l_1$ . From § 104

$$\text{Proj}_{l_2} OP = \text{Proj}_{l_2} OMNP.$$

$$\therefore OP \cos \theta = OM \cos \alpha_2 + MN \cos \beta_2 + NP \cos \gamma_2.$$

But,

$$OM = OP \cos \alpha_1, \quad MN = OP \cos \beta_1, \quad NP = OP \cos \gamma_1.$$

$$\therefore \cos \theta = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2,$$

$$(6) \quad \cos \theta = \lambda_1 \lambda_2 + \mu_1 \mu_2 + \nu_1 \nu_2.$$

We shall take  $\theta$  to be the smallest positive angle which satisfies (6).

**106. Perpendicular lines; parallel lines.** As a corollary to the last theorem we have

$$\lambda_1 \lambda_2 + \mu_1 \mu_2 + \nu_1 \nu_2 = 0$$

when and only when the lines are perpendicular. We call this the condition for perpendicularity.

Conditions for parallelism are either

$$\lambda_1 = \lambda_2, \quad \mu_1 = \mu_2 \text{ and } \nu_1 = \nu_2, \text{ or}$$

$$\lambda_1 = -\lambda_2, \quad \mu_1 = -\mu_2 \text{ and } \nu_1 = -\nu_2, \text{ or}$$

$$\lambda_1 \lambda_2 + \mu_1 \mu_2 + \nu_1 \nu_2 = 1, \text{ or}$$

$$\lambda_1 \lambda_2 + \mu_1 \mu_2 + \nu_1 \nu_2 = -1.$$

**107. Area of a triangle.** There are more or less elaborate formulas for the area of a triangle. However, a simple formula and one easily applied is the following:

$$A = \frac{1}{2} r_1 r_2 \sin \theta,$$

where  $r_1$  and  $r_2$  are the lengths of two sides and  $\theta$  is the angle between them. We have developed formulas for finding  $r_1$ ,  $r_2$  and  $\theta$  if the coordinates of the vertices are given.

### Exercises

1. By means of direction cosines prove that the points  $(5, 2, -3)$ ,  $(6, 1, 4)$ ,  $(-2, -3, 6)$  and  $(-1, -4, 13)$  are the vertices of a parallelogram.
2. By means of direction cosines prove that the points  $(6, -3, 5)$ ,  $(8, 2, 2)$  and  $(4, -8, 8)$  lie on a line.
3. Given points A  $(1, 4, 3)$ , B  $(-2, 7, -8)$ , C  $(2, 1, 7)$ . Find the cosine of the acute angle from A B to A C.
4. Find direction cosines of a line that is perpendicular to two lines whose direction cosines are proportional to 1, 2, 3 and -2, 1, 4 respectively.
5. Find the rectangular components of a force of 25 lb. acting in a direction making  $45^\circ$  with  $OX$  and  $60^\circ$  with  $OY$ .
6. Find the area of the triangle whose vertices are  $(0, 0, 0)$ ,  $(1, 1, 1)$ ,  $(0, 2, 3)$ .
7. Find the area of the triangle in Exercise 3.

**108. Spherical coordinates.** Any point  $P$  in space determines (Fig. 140) the radius vector  $OP = r$ , the angle  $\phi$  between the radius vector and the  $z$ -axis, and the angle  $\theta$  between the  $x$ -axis and the projection of the radius vector on the  $xy$ -plane. The quantities  $r$ ,  $\theta$ ,  $\phi$  are called the spherical coordinates of the point  $P$ , the angle  $\theta$  the **longitude** and the angle  $\phi$  is called the **colatitude**.

Conversely, any three numbers,  $r$ ,  $\theta$ ,  $\phi$  determine in space a point  $P$  whose spherical coordinates are  $(r, \theta, \phi)$ .

The equations of transformation from rectangular to spherical coordinates are

$$x = r \cos \theta \sin \phi,$$

$$y = r \sin \theta \sin \phi,$$

$$z = r \cos \phi.$$

The development of these formulas from Fig. 140 is left as an exercise.

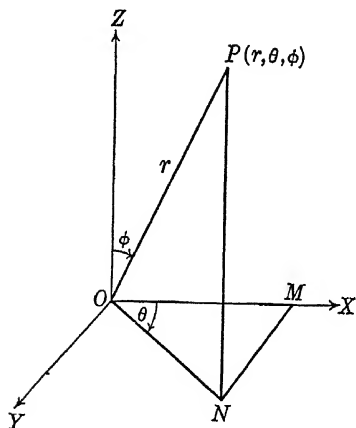


FIG. 140

**109. Cylindrical coordinates.** Let  $P'$  be the projection of  $P$  on the  $xy$ -plane. Let the polar coordinates of  $P'$  be  $(r, \theta)$  and  $z$  the distance of  $P$  from the  $xy$ -plane. The three numbers  $(r, \theta, z)$  are called the **cylindrical coordinates** of  $P$ . Conversely, any three numbers

$r$ ,  $\theta$ ,  $z$ , determine a point whose cylindrical coordinates are  $(r, \theta, z)$ .

The equations of transformation from rectangular to cylindrical coordinates are

$$x = r \cos \theta,$$

$$y = r \sin \theta,$$

$$z = z.$$

The development of these formulas from Fig. 141 is left as an exercise.

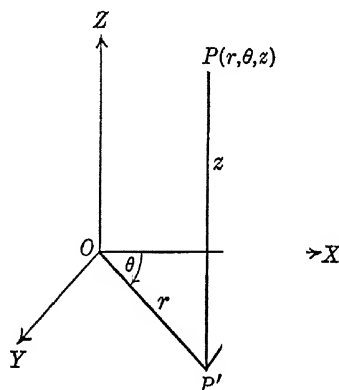


FIG. 141

### Exercises

- Find the rectangular coordinates of the points whose spherical coordinates are as follows:

$$\left(4, \frac{\pi}{2}, \frac{\pi}{4}\right); \quad \left(10, \frac{\pi}{4}, \frac{3\pi}{4}\right); \quad \left(10, -\frac{\pi}{4}, \frac{3\pi}{4}\right);$$

$$\left(10, -\frac{\pi}{3}, -\frac{3\pi}{4}\right).$$

2. Find the spherical coordinates of the points whose rectangular coordinates are  $(1, 1, 2)$ ;  $(1, 1, 1)$ ;  $(1, -2, 2)$ ;  $(2, 2, -2)$ .
3. Find the rectangular coordinates of the points whose cylindrical coordinates are  $(3, 30^\circ, 10)$ ;  $(4, 60^\circ, -2)$ ;  $(8, -150^\circ, 4)$ .
4. Find the cylindrical coordinates of the points whose rectangular coordinates are  $(1, 1, 2)$ ;  $(1, 1, -4)$ ;  $(2, -2, 2)$ .
5. Find a formula for the distance between any two points in terms of their spherical coordinates.
6. If the earth is considered a sphere with radius 3962 miles, what are the rectangular coordinates of a point on its surface in latitude  $40^\circ 21'$  North and longitude  $76^\circ 30'$  West? What is the distance from the polar axis?

## CHAPTER XIII

### THE PLANE — STRAIGHT LINE

**110. The locus of an equation.** The notion of the locus of an equation in three variables is entirely analogous to the notion of the locus of an equation in plane analytic geometry.

The locus of the equation consists of all those points and only those points whose coordinates satisfy the equation.

Similarly: An equation of a locus is an equation which is satisfied by the coordinates of all points of the locus and by the coordinates of no other point.

It is not necessary for all three variables to occur in an equation in order for us to speak of its locus in space. For example, the equation  $x = a$  is satisfied by the coordinates of all points whose  $x$ -coordinate is  $a$  and by no others. Hence  $x = a$  is the equation of the plane perpendicular to  $OX$  and a distance  $a$  from  $O$ , and is the locus of these points. We shall see later that  $Ax + By + C = 0$ , interpreted in three-dimensional geometry, is the equation of a plane parallel to the  $z$ -axis, etc.

**111. Theorem 1.** *Every plane has an equation of the first degree in  $x, y, z$ .*

Proof: Let  $S$  be any plane and let  $OR$  be the perpendicular from  $O$  meeting  $S$  in  $Q$ . The positive direction of

$OR$  will be taken from  $O$  to the plane. Let the direction angles of  $OR$  be  $\alpha, \beta, \gamma$  and let the length of  $OQ$  be  $p$ .\*

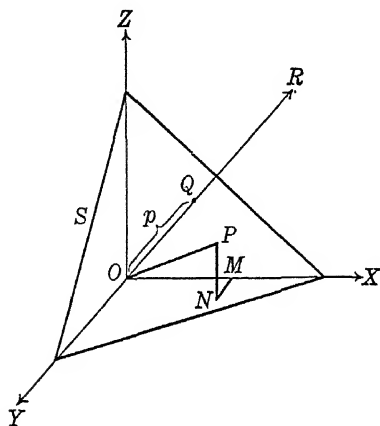


FIG. 142

If  $P(x, y, z)$  is any point in the plane, we have

$$\text{Proj}_{OR} OP = \text{Proj}_{OR} OM + \text{Proj}_{OR} MN + \text{Proj}_{OR} NP.$$

Hence the equation

$$(1) \quad x \cos \alpha + y \cos \beta + z \cos \gamma = p,$$

is the equation of the plane, which proves that any plane has an equation of the first degree.

Equation (1) is called the normal equation of the plane.

**Theorem 2.** *Every equation of the first degree in  $x, y$ , and  $z$  is the equation of a plane.*

Proof: Consider the equations

\* If the plane passes through  $O$  we shall take the positive direction of  $OR$  to be upward and hence  $\cos \gamma > 0$  since  $\gamma < 90^\circ$ . If the plane passes through the  $z$ -axis,  $OR$  lies in the  $xy$ -plane and  $\cos \gamma = 0$ . In this case we shall direct  $OR$  so that  $\beta < 90^\circ$  and hence  $\cos \beta > 0$ . If the plane coincides with the  $yz$ -plane, the positive direction on  $OR$  shall be taken as along  $OX$ .

$$(2) \quad Ax + By + Cz + D = 0$$

$$(1) \quad x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0.$$

Equation (1) represents a plane by Theorem 1. If equation (2) represents a plane it should differ from equation (1) only by a constant multiplier, say  $k$ .

Hence

$$kA = \cos \alpha, \quad kB = \cos \beta, \quad kC = \cos \gamma, \quad kD = -p.$$

$$\text{But} \quad \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1, \quad \S 103.$$

Therefore

$$k^2 A^2 + k^2 B^2 + k^2 C^2 = 1 \text{ or } k = \frac{1}{\pm \sqrt{A^2 + B^2 + C^2}}.$$

Hence equation (2) represents a plane in which

$$(3) \quad \cos \alpha = \frac{A}{\pm \sqrt{A^2 + B^2 + C^2}},$$

$$\cos \beta = \frac{B}{\pm \sqrt{A^2 + B^2 + C^2}},$$

$$\cos \gamma = \frac{-D}{\pm \sqrt{A^2 + B^2 + C^2}} \quad p = \frac{-D}{\pm \sqrt{A^2 + B^2 + C^2}}$$

**112. The angle between two planes.** The angle between two planes is the same as the angle between two lines perpendicular (normals) to the planes. If the equations of the planes are  $A_1x + B_1y + C_1z + D_1 = 0$ , and  $A_2x + B_2y + C_2z + D_2 = 0$ , the direction cosines of their normals are

$$\cos \alpha_1 = \frac{A_1}{\pm \sqrt{A_1^2 + B_1^2 + C_1^2}}, \quad \cos \alpha_2 = \frac{A_2}{\pm \sqrt{A_2^2 + B_2^2 + C_2^2}},$$

$$\begin{aligned}\cos \beta_1 &= \frac{B_1}{\pm \sqrt{A_1^2 + B_1^2 + C_1^2}} & \cos \beta_2 &= \frac{B_2}{\pm \sqrt{A_2^2 + B_2^2 + C_2^2}} \\ \cos \gamma_1 &= \frac{C_1}{\pm \sqrt{A_1^2 + B_1^2 + C_1^2}} & \cos \gamma_2 &= \frac{C_2}{\pm \sqrt{A_2^2 + B_2^2 + C_2^2}}\end{aligned}$$

If the angle between the normals is  $\theta$ , we have

$$(4) \quad \cos \theta = \pm \frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

If the planes are perpendicular,  $\cos \theta = 0$  and

$$(5) \quad A_1 A_2 + B_1 B_2 + C_1 C_2 = 0.$$

Conversely, if (5) is true, the planes are perpendicular.

**113. Parallel planes.** If two planes are parallel, their normals are parallel. Let the equations of the planes be

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p,$$

$$x \cos \alpha' + y \cos \beta' + z \cos \gamma' = p'.$$

Then either

$$\begin{aligned}\cos \alpha &= \cos \alpha', & \cos \beta &= \cos \beta', & \cos \gamma &= \cos \gamma', \text{ or} \\ \cos \alpha &= -\cos \alpha', & \cos \beta &= -\cos \beta', & \cos \gamma &= -\cos \gamma' .\end{aligned}$$

Therefore, if the equations of the two planes are

$$A_1 x + B_1 y + C_1 z + D_1 = 0,$$

$$A_2 x + B_2 y + C_2 z + D_2 = 0,$$

they will be parallel, if and only if

$$A_2 = kA_1, \quad B_2 = kB_1, \quad C_2 = kC_1. \quad (k \neq 0)$$

Therefore any plane parallel to  $Ax + By + Cz + D = 0$  can be written in the form  $Ax + By + Cz + k = 0$ .

*Example 1.* Find the equation of the plane passing through the points

$$(2, 1, 3), (1, 3, 2), (-1, 2, 4).$$

Solution: *Method I.* Let the equation of the plane be

$$Ax + By + Cz + D = 0.$$

Since the plane contains the three given points we have

$$2A + B + 3C + D = 0,$$

$$A + 3B + 2C + D = 0,$$

$$-A + 2B + 4C + D = 0.$$

$$\text{Solving, } A = -\frac{3}{25}D, \quad B = -\frac{4}{25}D, \quad C = -\frac{1}{5}D.$$

$$\text{Then} \quad -\frac{3}{25}Dx - \frac{4}{25}Dy + \frac{1}{5}Dz + D = 0,$$

$$\text{or} \quad 3x + 4y + 5z - 25 = 0,$$

is the equation of the plane through the three points.

*Method II.* If we eliminate  $A$ ,  $B$ ,  $C$ , and  $D$  from the four equations,

$$Ax + By + Cz + D = 0, \quad 2A + B + 3C + D = 0,$$

$$A + 3B + 2C + D = 0, \quad -A + 2B + 4C + D = 0,$$

we have

$$\begin{vmatrix} x & y & z \\ 2 & 1 & 3 \\ 1 & 3 & 2 \\ -1 & 2 & 4 \end{vmatrix} = 0,$$

or  $3x + 4y + 5z - 25 = 0$  as the equation of the plane. See Ex. 15, p. 272.

*Example 2.* Find the equation of the plane through  $(3, 2, 1)$  parallel to  $4x - 3y + z = 7$ .

Solution: The equation of any plane parallel to the given plane is

$$4x - 3y + z = k.$$

Substituting the coordinates of the point  $(3, 2, 1)$  gives  $k = 7$ .

Hence the equation of the required plane is

$$4x - 3y + z = 7.$$

*Example 3.* Find the equation of the plane passing through  $(3, -1, 2)$  and perpendicular to the planes  $2x - 3y + z = 4$  and  $x + 2y + 3z = 5$ .

Solution: Let the equation of the desired plane be

$$Ax + By + Cz + D = 0.$$

Since the plane passes through  $(3, -1, 2)$ , we have

$$3A - B + 2C + D = 0.$$

From the conditions for perpendicularity we have

$$2A - 3B + C = 0,$$

$$A + 2B + 3C = 0.$$

Solving we have

$$A = -\frac{11}{14}D, \quad B = -\frac{5}{14}D, \quad C = \frac{1}{2}D.$$

$$-\frac{11}{14}Dx - \frac{5}{14}Dy + \frac{1}{2}Dz + D = 0$$

or  $11x + 5y - 7z - 14 = 0$ ,

as the equation of the desired plane.

For a solution by determinants see Ex. 17, p. 272.

### Exercises

- Draw the planes whose equations are
  - $y = 3$ ,
  - $x = 3$ ,
  - $z = -2$ ,
  - $2x + y = 4$ ,
  - $4x + 3y + z = 12$ .
- What is the general equation of a plane passing through the origin?
- What is the equation of the  $xy$ -plane?  $yz$ -plane?  $xz$ -plane?
- What are the intercepts on the axes of the planes whose equations are:
  - $4x - 3y + z = 12$ ?
  - $x - y + z + 4 = 0$ ?
  - $x + y = 0$ ?
  - $Ax + By + Cz + D = 0$ ?
- Find three numbers proportional to the direction cosines of the normal to the plane  $3x + 4y - 5z = 6$ . What are the direction cosines?
- Find the equation of the plane through the points
  - $(1, 2, 3)$ ,  $(1, 0, 1)$ ,  $(2, 1, 0)$ ;
  - $(2, 2, 2)$ ,  $(2, 3, 1)$ ,  $(1, -1, 3)$ ;
  - $(3, 0, 1)$ ,  $(1, 0, 3)$ ,  $(2, 1, 2)$ .
- Find the equation of the plane through  $P$  and parallel to  $\alpha$  when
  - $P$  is  $(2, 1, 3)$  and  $\alpha$  is  $3x - 4y - 2z = 7$ ;
  - $P$  is  $(-1, 2, 3)$  and  $\alpha$  is  $2x + 3y + z = 5$ ;
  - $P$  is  $(1, -1, 2)$  and  $\alpha$  is  $5x - 3y + 7z = 6$ .

8. Find the equation of the plane passing through  $P_1$ ,  $P_2$  and perpendicular to  $\alpha$ , when
- $P_1$  is  $(1, 1, 1)$ ,  $P_2$  is  $(-2, 1, 3)$ ,  $\alpha$  is  $3x - 4y + z = 6$ ;
  - $P_1$  is  $(-1, 2, 3)$ ,  $P_2$  is  $(3, 1, -2)$ ,  $\alpha$  is  $2x + 5y - z = 7$ ;
  - $P_1$  is  $(3, 2, 1)$ ,  $P_2$  is  $(1, 0, 5)$ ,  $\alpha$  is  $4x - y = 3$ .
9. Find the equation of the plane passing through  $P$  and perpendicular to  $\alpha$  and  $\alpha_1$  when
- $P$  is  $(2, 1, 1)$ ,  $\alpha$  is  $3x - y + z = 4$ ,  $\alpha_1$  is  $2x + 3y - z = 5$ ;
  - $P$  is  $(-1, 2, 3)$ ,  $\alpha$  is  $2x + y - z = 0$ ,  $\alpha_1$  is  $3x - 2y + z = 7$ ;
  - $P$  is  $(-1, 0, 2)$ ,  $\alpha$  is  $3x + y - z = 4$ ,  $\alpha_1$  is  $4x - y + z = 7$ .
10. If the  $x$ ,  $y$  and  $z$  intercepts of a plane are  $a$ ,  $b$  and  $c$  respectively, prove the equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

11. Find the equation of the plane whose  $x$ ,  $y$  and  $z$  intercepts are respectively
- $3, 4, 5$ ;
  - $-2, 3, -7$ ;
  - $4, -4, 4$ .
12. Find the cosine of the smallest angle between the planes  $\alpha$  and  $\beta$  when
- $\alpha$  is  $4x + y - z = 6$ ,  $\beta$  is  $3x - 2y + z = 7$ ;
  - $\alpha$  is  $5x - y + 6z = 7$ ,  $\beta$  is  $2x - y - z = 3$ ;
  - $\alpha$  is  $3x + 2y - 5z = 4$ ,  $\beta$  is  $2x - 3y + 5z = 8$ .
13. Prove that the distance from the plane  $Ax + By + Cz + D = 0$  to the point  $(x_1, y_1, z_1)$  is
- $$\frac{Ax_1 + By_1 + Cz_1 + D}{\pm \sqrt{A^2 + B^2 + C^2}}$$

14. Find the distance from the plane  $\alpha$  to the point  $P$  when

a)  $\alpha$  is  $4x - 3y - z = 4$ ,  $P$  is  $(4, 2, 3)$ ;

b)  $\alpha$  is  $5x - 3y + 2z = 6$ ,  $P$  is  $(3, -1, 2)$ ;

c)  $\alpha$  is  $2x + y - z = 4$ ,  $P$  is  $(2, 3, 5)$ .

15. Prove that the equation of the plane passing through the points  $P_1(x_1, y_1, z_1)$ ,  $P_2(x_2, y_2, z_2)$ ,  $P_3(x_3, y_3, z_3)$  is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

16. Prove that the equation of the plane passing through the points  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$  and perpendicular to

$$A_1x + B_1y + C_1z + D_1 = 0 \text{ is}$$

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ A_1 & B_1 & C_1 & 0 \end{vmatrix} = 0.$$

17. Prove that the equation of the plane passing through  $(x_1, y_1, z_1)$  and perpendicular to the planes

$$A_1x + B_1y + C_1z + D_1 = 0, \quad A_2x + B_2y + C_2z + D_2 = 0 \text{ is}$$

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ A_1 & B_1 & C_1 & 0 \\ A_2 & B_2 & C_2 & 0 \end{vmatrix} = 0.$$

18. Prove that the planes

$$A_1x + B_1y + C_1z + D_1 = 0, \quad A_2x + B_2y + C_2z + D_2 = 0,$$

$$A_3x + B_3y + C_3z + D_3 = 0, \quad A_4x + B_4y + C_4z + D_4 = 0$$

meet in a common point, if and only if,

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} = 0.$$

**114. Simultaneous linear equations.** From algebra we know that three simultaneous linear equations in three unknowns have in general a single solution, but may have an infinite number of solutions or no solutions. Since each equation is the equation of a plane we can prove this statement geometrically for the three planes may assume the following positions.

*Case I. No two of the planes are parallel or coincident.*

a) The three planes may intersect in a single point; then there is a single solution of the three simultaneous equations.

b) The three planes may intersect in a line; then there is an infinite number of solutions.

c) The three planes may intersect so that the three lines of intersection are parallel; then there is no solution.

*Case II. Two of the planes are parallel but not coincident.*

In this case the three planes can have no point in common and the equations have no solution.

*Case III. Two of the planes are coincident.*

a) The third plane may be parallel to the coincident planes, in which case there is no solution.

b) The third plane may intersect the coincident planes, in which case there is an infinite number of solutions.

c) The third plane may coincide with the coincident planes, in which case there is an infinite number of solutions.

**115. Pencil of planes.** All those planes which have a line in common are said to form a pencil of planes. The line in which they intersect is called the axis of the pencil.

If

$$(6) \quad A_1x + B_1y + C_1z + D_1 = 0$$

and

$$(7) \quad A_2x + B_2y + C_2z + D_2 = 0$$

are the equations of any two distinct planes of the pencil, then

(8)

$$(A_1x + B_1y + C_1z + D_1) + k(A_2x + B_2y + C_2z + D_2) = 0,$$

is the equation of any other plane passing through the given line.

**116. Sheaf or bundle of planes.** All the planes which pass through a common point are said to form a sheaf or bundle. The common point is called the center.

*Let the following be the equations of three distinct planes of the sheaf not belonging to the same pencil:*

$$A_1x + B_1y + C_1z + D_1 = 0,$$

$$A_2x + B_2y + C_2z + D_2 = 0,$$

$$A_3x + B_3y + C_3z + D_3 = 0.$$

Then the equation of the sheaf or bundle is

(9)

$$(A_1x + B_1y + C_1z + D_1) + k_1(A_2x + B_2y + C_2z + D_2) + k_2(A_3x + B_3y + C_3z + D_3) = 0,$$

where  $k_1$  and  $k_2$  are constants whose values determine the position of the particular plane of the sheaf or bundle.

### 117. Equation factorable into first degree factors.

The locus of the equation

$$(10) \quad (A_1x + B_1y + C_1z + D_1)(A_2x + B_2y + C_2z + D_2) \dots (A_nx + B_ny + C_nz + D_n) = 0$$

is the  $n$  planes whose equations are

$$A_1x + B_1y + C_1z + D_1 = 0,$$

$$A_2x + B_2y + C_2z + D_2 = 0,$$

$$\dots\dots\dots$$

$$A_nx + B_ny + C_nz + D_n = 0.$$

This follows immediately from the fact that a product is zero when and only when at least one of the factors of the product is zero. There is no reason why some of the factors of (10) should not be repeated. We then speak of coincident planes in exactly the same sense in which we speak of coincident roots of an equation.

### Exercises

1. Find the equation of the plane that passes through the intersection of  $\alpha$  and  $\beta$  and the point  $P$  when
  - a)  $\alpha$  is  $3x - 2y - z = 1$ ,  $\beta$  is  $4x + 3y - z = 4$ ,  $P$  is  $(1, 1, 1)$ ;

b)  $\alpha$  is  $4x + y - z = 3$ ,  $\beta$  is  $3x + 2y - z = 4$ ,  $P$  is  $(3, 2, 1)$ ;

c)  $\alpha$  is  $7x - y - z = 4$ ,  $\beta$  is  $2x + y - z = 4$ ,  $P$  is  $(2, 1, 3)$ .

2. Show that the planes whose equations are  $3x - 5y + 2z = 0$ ,  $6x + y - 2z - 13 = 0$ ,  $11y - 2z = 17$  belong to the same pencil.

3. Find the equation of the plane through the intersection of  $4x - 3y - z = 4$ ,  $5x + y - 2z = 7$  and perpendicular to the  $xy$ -plane.

4. Find the equation of the plane through the intersection of the planes  $\alpha$ ,  $\beta$ ,  $\gamma$  and the points  $P_1$ ,  $P_2$  when

a)  $\alpha$  is  $3x + y - z = 3$ ,  $\beta$  is  $2x - y - z = 0$ ,  
 $\gamma$  is  $4x - y - z = 2$ ,  $P_1$  is  $(2, 1, 3)$ ,  $P_2$  is  $(-2, 2, 1)$ .

b)  $\alpha$  is  $2x + y - z = 0$ ,  $\beta$  is  $3x + y + z = 2$ ,  
 $\gamma$  is  $4x - 3y - 2z = 4$ ,  $P_1$  is  $(2, 1, 1)$ ,  $P_2$  is  $(-1, 2, 1)$ .

### 118. Equations of a straight line.

a) The two simultaneous equations

$$A_1x + B_1y + C_1z + D_1 = 0,$$

$$A_2x + B_2y + C_2z + D_2 = 0,$$

have a line as locus, namely, their line of intersection, provided, of course, the two planes are not parallel or coincident.

b) A given point and a given direction determine a line. Let the given point be  $P_1(x_1, y_1, z_1)$  and  $\alpha$ ,  $\beta$ ,  $\gamma$  the given direction angles. If  $P(x, y, z)$  is any other point on the line, at a distance  $d$  from  $P_1$ , we have from § 103,

$$\cos \alpha = \frac{x - x_1}{d}, \quad \cos \beta = \frac{y - y_1}{d}, \quad \cos \gamma = \frac{z - z_1}{d}.$$

Therefore we have

$$(11) \quad \frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\cos \beta} = \frac{z - z_1}{\cos \gamma},$$

or 
$$\frac{x - x_1}{\lambda} = \frac{y - y_1}{\mu} = \frac{z - z_1}{\nu},$$

which are called the **symmetric equations** of a straight line. In these equations  $\cos \alpha$ ,  $\cos \beta$ ,  $\cos \gamma$  can of course be replaced by any three numbers proportional to them.

The equations of the line are

(12)

c) Two distinct points  $P_1(x_1, y_1, z_1)$ ,  $P_2(x_2, y_2, z_2)$  determine a line. Any line through  $P_1$  is of the form

$$\frac{x - x_1}{a}$$

But the direction cosines of  $P_1P_2$  are proportional to  $x_2 - x_1$ ,  $y_2 - y_1$ ,  $z_2 - z_1$ . Hence the equations of the line through the two points  $P_1$  and  $P_2$  are

$$(13) \quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1},$$

if denominator is zero.

d) If  $t$  denotes the variable distance of  $P_1P$ , then

$$\frac{x - x_1}{t} = \lambda, \quad \frac{y - y_1}{t} = \mu, \quad \frac{z - z_1}{t} = \nu.$$

Therefore

$$x = x_1 + \lambda t$$

$$(14) \quad \begin{aligned} y &= y_1 + \mu t \\ z &= z_1 + \nu t, \end{aligned}$$

which are parametric equations of the line through the point  $(x_1, y_1, z_1)$  with direction cosines  $\lambda, \mu, \nu$ .

It should be noted that if the line is parallel to one of the coordinate axes, forms (11), (12) and (13) cannot be used, for some one of the denominators would be zero. However, form (14) is valid under all conditions.

*Example 1.* Reduce to the symmetric form the equations of the straight line,

$$3x + 2y - z = 4, \quad x + 3y + z = 5.$$

Solution: Eliminating  $z$  we have  $4x + 5y = 9$ . Similarly, eliminating  $x$  we have  $7y + 4z = 11$ . Solving these equations for  $y$  and equating the values found, we have

$$\frac{y}{1} = \frac{4x - 9}{-5} = \frac{4z - 11}{-7}$$

or

$$\frac{x - \frac{9}{4}}{-\frac{5}{4}} = \frac{y - 0}{1} = \frac{z - \frac{11}{4}}{-\frac{7}{4}}$$

Hence the line passes through the point  $(\frac{9}{4}, 0, \frac{11}{4})$  and has direction cosines proportional to  $-5, 4, -7$ .

*Example 2.* Find the equations of the line passing through  $(3, 4, -5)$  and perpendicular to the plane  $5x - 2y + 7z = 5$ .

Solution: The required line is parallel to the perpendicular from the origin to the given plane and

therefore the direction cosines are proportional to 5, -2, 7. The required line then has the equations,

$$\frac{x-3}{5} = \frac{y-4}{-2} = \frac{z+5}{7}.$$

### Exercises

1. Write the equations of the line passing through point  $P$  with direction cosines proportional to  $a, b, c$ , when:
  - a)  $P$  is (3, 4, 1),  $a = 5$ ,  $b = 2$ ,  $c = -2$ .
  - b)  $P$  is (-2, 4, -3),  $a = 2$ ,  $b = 3$ ,  $c = -5$ .
  - c)  $P$  is (2, 0, 1),  $a = -1$ ,  $b = 4$ ,  $c = 6$ .
2. Find the equations of the lines passing through the following pairs of points:
  - a) (3, 4, 5), (2, 5, -1).
  - b) (4, 6, 5), (2, -1, 4).
  - c) (-2, 3, 2), (4, 4, 2).
3. Write in symmetric form the equations of the lines:
  - a)  $3x + 2y - z = 4$ ,  $2x - y + 3z = 5$ .
  - b)  $3x - y + z = 4$ ,  $x + y + z = 4$ .
  - c)  $x - 2y - 7z = 3$ ,  $2x + y + z = 3$ .
4. Find the equations of the line that passes through the point  $P$  and is perpendicular to plane  $\alpha$ , when:
  - a)  $P$  is (2, -1, 3),  $\alpha$  is  $4x - 3y + 2z = 4$ .
  - b)  $P$  is (1, 0, 2),  $\alpha$  is  $3x - 2y - z = 2$ .
  - c)  $P$  is (2, -1, 0),  $\alpha$  is  $5x + 2y - 7z = -3$ .

5. Find the equations of the line that passes through the point  $(4, -1, 3)$  and is parallel to the line

$$\frac{x-7}{3} = \frac{y+2}{5} = \frac{z}{-7}.$$

6. Find in symmetric form the equations of the line that passes through  $(7, -2, 4)$  and is parallel to the line

$$4x - 3y - z = 1, \quad 2x + 4y + z = 5.$$

7. Find the equation of the plane through  $(2, -1, 4)$  and perpendicular to the line

$$\frac{x-7}{4}, \quad \frac{y-3}{-2} = \frac{z+1}{5}.$$

8. Prove that the lines

$$\frac{x}{6} = \frac{y}{-2} = \frac{z}{-4} \quad \text{and} \quad \frac{x}{4} = \frac{y}{6} = \frac{z}{3}$$

are perpendicular to each other.

### MISCELLANEOUS EXERCISES

1. Find the intercepts of the planes which have the following equations:

a)  $3x + 2y - 5z = 60.$

b)  $7x - 8y + 3z = 2.$

c)  $x \cos 60^\circ + y \sin 60^\circ + z \cos 45^\circ = 10.$

d)  $2x - 3y = 4.$

2. What are the equations of the planes whose  $x$ ,  $y$ , and  $z$  intercepts are respectively

a)  $1, 4, 5;$  b)  $3, -6, 2;$  c)  $4, -1, -1.$

3. Describe in words the location of the planes which have the following equations:

$$a) \frac{x}{1} + \frac{y}{-2} + \frac{z}{3} = 1. \quad b) 2x + 3y + 4z = 1.$$

$$c) x - y = 2. \quad d) z + y = -4. \quad e) y = 1.$$

4. Find the equation of the plane passing through the points

$$(3, -1, 4), \quad (2, 1, 5), \quad (-1, 3, 2).$$

5. Show that the following four points lie in a plane:

$$(1, 1, -11), \quad (5, 0, 9), \quad (5, -5, 25), \quad (0, 0, -12).$$

6. What are the direction cosines of the normals to the plane

$$4x + 5y - 3z + 2 = 0?$$

What is the distance of this plane from the origin?

7. What is the perpendicular distance of the point  $(-1, 2, 3)$  from the plane whose equation is

$$6x + 5y - z = 8?$$

8. Are the points  $(6, 1, -4)$  and  $(4, -2, 3)$  on the same side of the plane

$$2x + 3y - 5z + 1 = 0?$$

9. Find the equation of the plane through  $(6, 2, 4)$  which is perpendicular to the line drawn from the origin to that point.

10. What is the equation of the plane through the point  $(1, -2, 4)$  parallel to the plane whose equation is

$$6x + y - 4z + 2 = 0?$$

11. Which pair of the following planes are perpendicular?

a)  $7x - 8y + 3z + 2 = 0$ ,

b)  $2x - 3y + 3z + 10 = 0$ ,

c)  $3x + y - z + 4 = 0$ .

12. Reduce the equations of the following planes to the intercept form and to the normal form:

a)  $7x + 6y - z = 4$ .

b)  $8x - 10y + 3z = 8$ .

13. Write the equations of the planes equally inclined to the axes and at a distance 4 from the origin.

14. What is the cosine of the acute angle between the planes whose equations are:

$$4x + 5y - 6z + 1 = 0, \quad x + y + z + 4 = 0.$$

15. Find the equation of the plane through  $(1, 1, 1)$  and  $(-1, 2, 3)$  and perpendicular to the plane whose equation is  $4x - 5y + 2z = 12$ .

16. If  $a, b, c$  are the intercepts of a plane and  $p$  its distance from the origin, prove

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}.$$

17. Derive a formula for the equation of the plane which is the perpendicular bisector of the line segment joining  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$ . Apply your formula to a numerical example of your own choosing.

18. If  $\theta$  is the angle between the two lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \quad \text{and} \quad \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2},$$

find  $\cos \theta$ .

19. A plane passes through the line of intersection of the planes whose equations are:

$$2x - 5y + 11z = 4 \text{ and } x - 6y + z = -3,$$

and through the point  $(1, 0, 1)$ . What is its equation?

20. A plane passes through the point of intersection of the planes whose equations are:

$$2x - 5y + 11z = 4,$$

$$x - 6y + z = -3,$$

$$x + y + z = 1,$$

and through the two points  $(1, 4, 2)$  and  $(-1, 1, -1)$ . What is its equation?

21. What is the locus of:

a)  $x^2 - 5x + 6 = 0$ ?

b)  $x^2 - y^2 = 0$ ?

c)  $x^2 + 2xy + y^2 - z^2 - 2z - 1 = 0$ ?

d)  $y = x^2$ ?

22. Do the planes which have the following equations pass through the same straight line?

$$2x + 5y + 3z = 0.$$

$$7y - 5z + 4 = 0.$$

$$x - y + 4z - 2 = 0.$$

23. If  $\lambda, \mu, \nu$  are the direction cosines of the line of intersection of  $A_1x + B_1y + C_1z + D_1 = 0$ ,  $A_2x + B_2y + C_2z + D_2 = 0$  prove that

$$\begin{array}{ccc} \lambda & \mu & \nu \\ \left| \begin{array}{c} B_1C_1 \\ B_2C_2 \end{array} \right| & \left| \begin{array}{c} C_1A_1 \\ C_2A_2 \end{array} \right| & \left| \begin{array}{c} A_1B_1 \\ A_2B_2 \end{array} \right| \end{array}$$

24. Prove that the equation of the plane through the point  $(x_1, y_1, z_1)$  and parallel to the two lines whose direction cosines are  $\lambda_1, \mu_1, \nu_1$  and  $\lambda_2, \mu_2, \nu_2$  respectively is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \end{vmatrix} = 0.$$

25. Determine if the following planes pass through a point:

$$\begin{aligned} x + 2y - z + 3 &= 0, \\ 3x - y + 2z + 1 &= 0, \\ 2x - 2y + 3z - 2 &= 0, \\ x - y - z + 3 &= 0. \end{aligned}$$

26. Find the values of  $k$  for which the following planes are perpendicular:

$$kx + 3y + kz = 4 \text{ and } (k - 1)x + ky + z = 0.$$

27. For what values of  $D$  do the planes with the following equations meet in a point?

$$\begin{aligned} Dx + 2y - z + 7 &= 0, \\ 3x - y + 2z + 1 &= 0, \\ 2x - y + z - 2 &= 0, \\ x + y - z + D &= 0. \end{aligned}$$

## CHAPTER XIV

### QUADRIC SURFACES

**119.** A quadric surface is any surface whose equation is of the second degree in  $x$ ,  $y$ , and  $z$ . A quadric surface is sometimes called a conicoid.

A sphere is the locus of a point whose distance from a fixed point is constant. The fixed point is called the center and the fixed distance the radius of the sphere.

The term radius is also applied to any directed line segment from the center to a point of the sphere as well as to the length of that segment.

Let the center be  $C(h, k, l)$  and  $P(x, y, z)$  any point on the sphere. By the definition of a sphere  $CP = r$ ,

or 
$$\overline{CP}^2 = r^2.$$

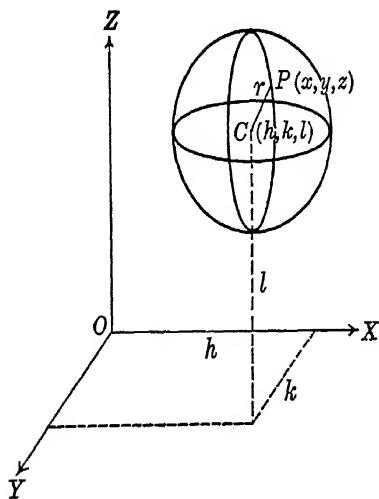


FIG. 143

$$(1) \quad \therefore (x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

If the center is at the origin, (1) reduces to

$$(2) \quad x^2 + y^2 + z^2 = r^2.$$

If the sphere passes through the origin and has its center on the  $X$ -axis, (1) reduces to

$$(3) \quad x^2 - 2rx + y^2 + z^2 = 0.$$

If equation (1) is expanded it will take the form

$$(4) \quad x^2 + y^2 + z^2 + ax + by + cz + d = 0,$$

where  $a, b, c$ , and  $d$  are constants. Conversely, equation (4) can be written

$$(5) \quad (x + a/2)^2 + (y + b/2)^2 + (z + c/2)^2 = \frac{a^2 + b^2 + c^2 - 4d}{4}.$$

If  $a^2 + b^2 + c^2 - 4d > 0$ , (5) is the equation of a sphere with center at  $(-a/2, -b/2, -c/2)$  and radius  $\frac{1}{2}\sqrt{a^2 + b^2 + c^2 - 4d}$ . If  $a^2 + b^2 + c^2 - 4d = 0$ , equation (4) is satisfied by the coordinates of one point only, namely  $(-a/2, -b/2, -c/2)$ . If

$$a^2 + b^2 + c^2 - 4d < 0$$

there is no point whose coordinates satisfy the equation.

**120. Essential constants.** Equations (1) and (4) each contains four constants. A sphere can usually be made to fulfill four given conditions by properly choosing these constants. The student must be on his guard, however, against exceptional circumstances when there is no sphere to fulfill the given conditions of the problem. For example, to find the equation of the sphere through

the four points  $P_1 (x_1, y_1, z_1)$ ,  $P_2 (x_2, y_2, z_2)$ ,  $P_3 (x_3, y_3, z_3)$ , and  $P_4 (x_4, y_4, z_4)$  substitute the coordinates of the points successively in (4) and solve the resulting equations for  $a$ ,  $b$ ,  $c$ , and  $d$ . This is a possible process in general, although in exceptional cases the equations may be inconsistent, as will appear when their solution is attempted. Such a case will arise if the four points lie on a line.

*Example.* Find the equation of the sphere through  $(1, 1, 1)$ ,  $(1, 2, 1)$ ,  $(1, 1, 2)$ , and  $(2, 1, 1)$ .

Solution: The coordinates of the given points must satisfy the equation

$$x^2 + y^2 + z^2 + ax + by + cz + d = 0.$$

This gives the equations

$$a + b + c + d = -3,$$

$$a + 2b + c + d = -6,$$

$$a + b + 2c + d = -6,$$

$$2a + b + c + d = -6.$$

Solving these equations we have  $a = -3$ ,  $b = -3$ ,  $c = -3$ ,  $d = 6$ . Hence the equation of the sphere is

$$x^2 + y^2 + z^2 - 3x - 3y - 3z + 6 = 0.$$

This problem can be solved also by determinants. See Ex. 6, p. 288.

### Exercises

- Find the equations of the spheres which have the following points as centers and which have the indicated radii:

$$a) (-1, 3, 2), r = 4; \quad b) (7, 8, -6), r = 2;$$

$$c) (4, 3, -2), r = 10; \quad d) (0, 0, 1), r = 1.$$

2. Determine which of the following equations is the equation of a sphere. For each sphere determine the coordinates of the center and the length of the radius:

$$a) x^2 + y^2 + z^2 + 2x + 4y + 6z + 20 = 0.$$

$$b) x^2 + y^2 + z^2 + 2x + 4y + 6z + 1 = 0.$$

$$c) x^2 + 2x + y^2 + z^2 = 0.$$

$$d) x^2 + y^2 + z^2 + 2x + 2y = 0.$$

$$e) x^2 + y^2 + z^2 + 2x + 4y + 6z + 14 = 0.$$

$$f) 3x^2 + 3y^2 + 3z^2 + 5x - 6y + 7z = 0.$$

$$g) 3x^2 + 2y^2 + 4z^2 - x + y - z + 2 = 0.$$

3. Find the equation of the sphere which has the points  $(-1, 3, 2)$  and  $(6, 7, 8)$  as ends of a diameter.
4. Find the equations of the spheres which have a radius  $r$  and which are tangent to all the coordinate planes.
5. Find the equation of the sphere which has its center at  $(1, 1, 1)$  and passes through  $(-1, 4, 2)$ .
6. If  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$ ,  $(x_3, y_3, z_3)$ ,  $(x_4, y_4, z_4)$  are the vertices of a tetrahedron, prove that the equation of the circumscribed sphere is

$$\begin{vmatrix} (x^2 + y^2 + z^2) & x & y & z & 1 \\ (x_1^2 + y_1^2 + z_1^2) & x_1 & y_1 & z_1 & 1 \\ (x_2^2 + y_2^2 + z_2^2) & x_2 & y_2 & z_2 & 1 \\ (x_3^2 + y_3^2 + z_3^2) & x_3 & y_3 & z_3 & 1 \\ (x_4^2 + y_4^2 + z_4^2) & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

7. When possible find the equations of the spheres that pass through the following points:

a)  $(1, 1, 1), (0, 1, 0), (2, 0, 0), (0, 0, 2);$

b)  $(1, 1, 1), (0, 0, 0), (4, -1, 3), (-1, -1, -1);$

c)  $(5, -1, 2), (3, 1, 4), (7, 8, 6), (0, 3, 1);$

d)  $(1, 1, 9), (1, 2, 7), (2, 5, 3), (-1, 3, 8).$

8. Show that the equation of the plane tangent to the sphere  $x^2 + y^2 + z^2 = r^2$  at point  $(x_1, y_1, z_1)$  is

$$xx_1 + yy_1 + zz_1 = r^2.$$

[Hint: The tangent plane is perpendicular to the radius drawn to the given point.]

9. Show that the equation of the plane tangent to

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2 \text{ at point } P_1 (x_1, y_1, z_1)$$

$$\text{is } (x_1 - h)(x - x_1) + (y_1 - k)(y - y_1) + (z_1 - l)(z - z_1) = 0.$$

**121. Cylinders.** A surface generated by a straight line moving parallel to a given line and always intersecting a given fixed curve is called a **cylindrical surface** or a **cylinder**. An **element** of the cylinder is the generating line in any of its positions.

Any algebraic equation in two cartesian coordinates is the equation in three-dimensional geometry of a cylinder whose elements are parallel to the axis of the third variable while in two-dimensional geometry it represents a curve. For example,  $x^2 + y^2 = 9$  in two-dimensional geometry is the equation of a circle in the  $xy$ -plane. However, in three-dimensional geometry the equation is satisfied by the coordinates of any point  $P$  situated

on a line meeting the circle at  $M$  and parallel to the  $z$ -axis. Moreover, if  $MP$  moves parallel to the  $z$ -axis

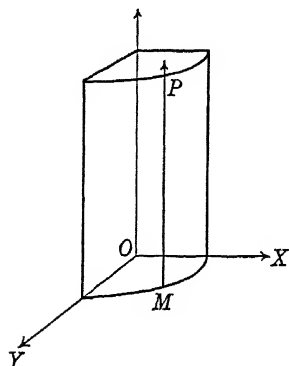


FIG. 144

and continues to intersect the circle, the coordinates of  $P$  continue to satisfy the equation  $x^2 + y^2 = 9$ . Hence the line  $MP$  traces a cylinder which is the locus of the equation  $x^2 + y^2 = 9$ .

If a plane is passed perpendicular to the axis of a cylinder, the section is a curve equal and parallel to the directing curve.

If a cylinder has its elements parallel to one of the coordinate axes and always intersects a fixed curve in space, the cylinder is called the **projecting cylinder** of the curve.

### Exercises

Describe the locus of each of the following equations:

1.  $x = 5$ .
2.  $x^2 + y^2 = 4$ .
3.  $y = 2x$ .
4.  $x^2 - y^2 = 4$ .
5.  $y^2 = 4x$ .
6.  $4x^2 + y^2 = 36$ .

7.  $y^2 = x - 1$ .      8.  $z = 4x^2$ .      9.  $z^2 = y^2 + 1$ .  
 10.  $x^2 + z^2 = 16$ .    11.  $x^2 - z^2 = 0$ .    12.  $x^3 + xy^2 = 4$ .

**122. Symmetry, intercepts, traces, sections.** The following rules for symmetry are easily proved.

If a given equation is left unchanged by replacing  $x, (y), z$ , by  $-x, (-y), -z$ , the locus is by definition symmetric with respect to the  $yz, (xz), xy$ , plane respectively.

**Intercepts** of a surface on the axes are the segments from the origin to where the surface cuts the axes. To find the intercepts, place two of the variables equal to zero and solve the resulting equation for the third variable.

**Traces** of a surface are sections of the surface made by the coordinate planes. To find the equations of the

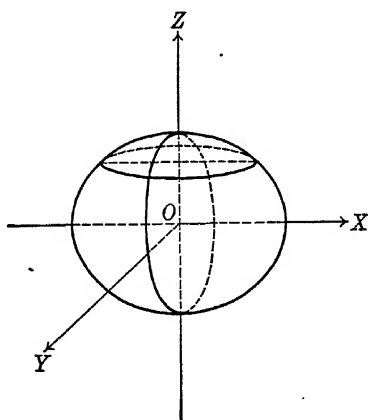


FIG. 145

traces, place successively each variable in the equation of the surface, equal to zero.

A **section** of a surface is the curve formed by the surface and a plane cutting it. For example, the equation of the surface and  $x = k$ , where  $k$  is a constant, are together a section, and the equations of the curve of intersection of the surface and a plane parallel to the  $yz$ -plane.

**123. The ellipsoid.** The locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

is called an **ellipsoid**. It is symmetric with reference to each of the coordinate planes and hence to the origin which is consequently called its **center**. Its intercepts on the positive directions of the axes are  $a$ ,  $b$ ,  $c$ , which are called its **semi-axes**. Sections made by any plane parallel to the coordinate axes are ellipses. For example, the section made by the plane  $z = k$  if the plane cuts the ellipsoid, has the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}, \quad z = k.$$

This is an ellipse if  $k^2 < c^2$ , a point if  $k^2 = c^2$ , and there is no locus if  $k^2 > c^2$  in which case the plane does not intersect the ellipsoid.

It may happen that all of the semi-axes are equal, *i.e.*,  $a = b = c$ , in which case the ellipsoid is a sphere. If two of the semi-axes are equal, for example if  $b = c$ , the ellipsoid is called an **ellipsoid of revolution** for it can be generated by revolving the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ,  $z = 0$ , about the  $x$ -axis.

**124. Surfaces of revolution.** The surface generated by revolving a plane curve about any line in its plane, is called a **surface of revolution**. When the axis of revolution, *i.e.*, the line about which the curve is revolved, is one of the coordinate axes, the equation of the surface is readily found.

*Example.* The parabola  $y^2 = 4x$ ,  $z = 0$  is revolved about the  $x$ -axis. Find the equation of the surface of revolution.

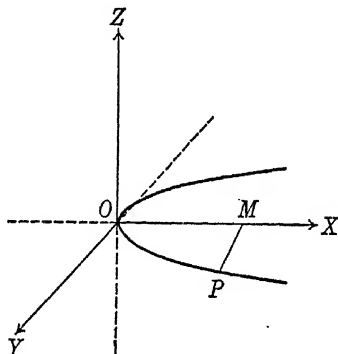


FIG. 146

**Solution:** As the curve is revolved about the  $x$ -axis any point  $P$  on this curve describes a circle, whose center is on the  $x$ -axis and whose radius is the ordinate  $MP$ . Therefore, for any position of  $P$ ,

$$y^2 + z^2 = \overline{MP}^2.$$

But  $\overline{MP}^2 = y^2 = 4x.$

$\therefore y^2 + z^2 = 4x$ , is the equation of the surface.

## Exercises

Find the equations of the surfaces generated by revolving the following curves about the indicated axes.

1.  $4x^2 + 9y^2 = 36$ ,  $z = 0$ ,  $x$ -axis.
2.  $4x^2 + 9y^2 = 36$ ,  $z = 0$ ,  $y$ -axis.
3.  $4x^2 + 9z^2 = 36$ ,  $y = 0$ ,  $x$ -axis.
4.  $4x^2 + 9z^2 = 36$ ,  $y = 0$ ,  $z$ -axis.
5.  $y^2 = 4z$ ,  $x = 0$ ,  $z$ -axis.
6.  $x^2 = 4y$ ,  $z = 0$ ,  $y$ -axis.
7.  $x^2 - y^2 = 4$ ,  $z = 0$ ,  $x$ -axis.
8.  $y^2 - z^2 = 16$ ,  $x = 0$ ,  $y$ -axis.
9.  $z^2 = 2x$ ,  $y = 0$ ,  $x$ -axis.
10.  $y^2 + x^2 = 4$ ,  $z = 0$ ,  $y$ -axis.
11.  $y = x^3$ ,  $z = 0$ ,  $x$ -axis.
12.  $z = 3x$ ,  $y = 0$ ,  $x$ -axis.
13. When an ellipse is revolved about its major axis the ellipsoid generated is called a **prolate spheroid**; when it is revolved about its minor axis, an **oblate spheroid**. Which of the spheroids in Examples 1, 2, 3, 4 are prolate? oblate?
14. Describe the locus of

$$a) \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0.$$

$$b) \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + 1 = 0.$$

**125. Elliptic paraboloid.** The locus of the equation

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

is called an **elliptic paraboloid**. It is evident that sections parallel to and above the  $xy$ -plane are ellipses, and that sections parallel to the  $xz$ -plane or the  $yz$ -plane are parabolas.

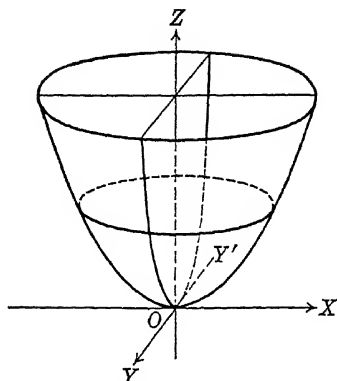


FIG. 147

**126. Elliptic hyperboloid of one sheet.** The locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

is called an **elliptic hyperboloid of one sheet**. Sections made by planes parallel to the  $xy$ -plane are ellipses, parallel to either of the other coordinate planes are hyperbolas.

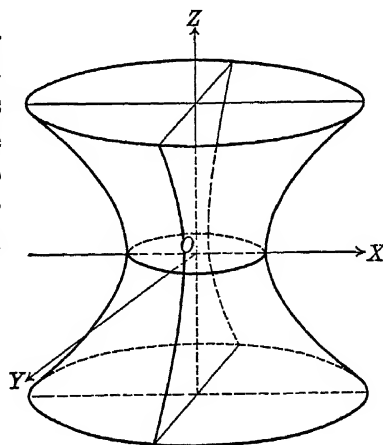


FIG. 148

127. Elliptic hyperboloid of two sheets. The locus of the equation

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is called an elliptic hyperboloid of two sheets. We can rewrite the equation in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} - 1,$$

whereupon we observe that sections made by the planes  $z = k$  are ellipses if  $c^2 < k^2$ , a point if  $c^2 = k^2$  and that there is no section if  $c^2 > k^2$ .

In a similar way we see that sections made by planes parallel to the  $xz$ -plane and to the  $yz$ -plane are hyperbolas.

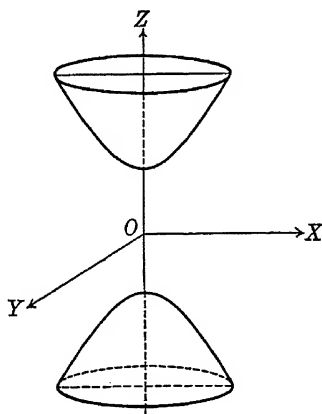


FIG. 149

128. The hyperbolic paraboloid. The locus of the equation

$$z = \frac{x^2}{a^2} - y^2$$

is called a hyperbolic paraboloid. Sections made by the planes  $z = k$  are hyperbolas. The asymptotes of each of these hyperbolas project orthogonally into the same two lines in the  $xy$ -plane. Sections parallel to the  $xz$ -plane and  $yz$ -plane are parabolas.

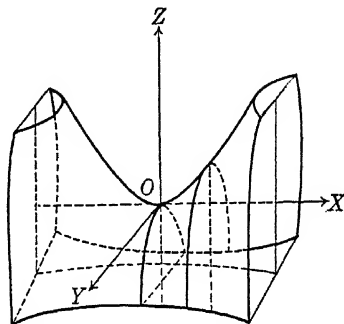


FIG. 150

129. The cone. The locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

is called an elliptic cone. It is symmetric with respect to the three coordinate planes, the three axes and the origin. All three of its intercepts are zero. Sections parallel to the  $xy$ -plane are ellipses, while those parallel to the  $xz$  and  $yz$  planes are hyperbolas. The traces on the  $xz$  and  $yz$  planes are respectively the pairs of lines  $cx \pm az = 0$ ,  $y = 0$ ;  $cy \pm bz = 0$ ,  $x = 0$ . If any

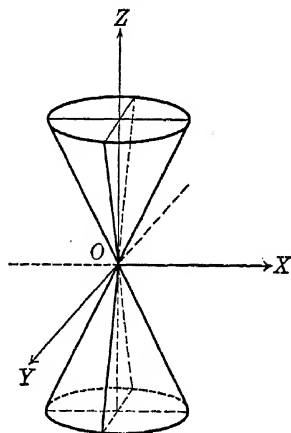


FIG. 151

point  $P(x_1, y_1, z_1)$  is on the surface and is connected to the origin  $O$ , then the line  $OP$  lies on the surface, for the coordinates of any point on the line are  $(\lambda x_1, \lambda y_1, \lambda z_1)$  and they are seen to satisfy the given equation.

If  $a = b$ , the cone is a cone of revolution.

**130. Ruled surfaces.** A surface which is such that through every one of its points there is a straight line which lies entirely on the surface, is called a **ruled surface**.

Examples with which the student is already familiar are planes, cylinders, and cones. It happens, however, that *two* of the quadric surfaces are also ruled surfaces.

Consider the equation of the elliptic hyperboloid of one sheet, namely, equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \quad \text{This can be written}$$

$$(6) \quad \left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right).$$

Now consider the family of lines whose equations are

$$(7) \quad \begin{aligned} \frac{x}{a} + \frac{z}{c} &= t \left(1 + \frac{y}{b}\right), \\ t \left(\frac{x}{a} - \frac{z}{c}\right) &= 1 - \frac{y}{b}. \end{aligned}$$

where  $t$  is a parameter. By varying  $t$  we get not one but a system or family of lines. The family of lines obtained

by giving all real values to  $t$  plus the line whose equations are

$$(8) \quad \begin{aligned} \frac{x}{a} - \frac{z}{c} &= 0, \\ 1 + \frac{y}{b} &= 0, \end{aligned}$$

will be called the completed family.

Each member of this completed family lies on the surface. For if equations (7) or (8) are satisfied it is evident that (6) is satisfied.

We shall also show that one member of this completed family passes through each point of the surface. Consider a point  $P(x, y, z)$  of the surface. If  $1 + \frac{y}{b} \neq 0$  determine  $t$  so as to satisfy the first of (7), substitute in (6) and we obtain the second of (7), that is, both planes whose equations are (7) pass through the point. If  $1 + \frac{y}{b} = 0$ , but  $\frac{x}{a} - \frac{z}{c} \neq 0$ , determine  $t$  to satisfy the second of (7). Substitution in (6) now yields the first of (7). Again both planes pass through the point. If  $\frac{x}{a} - \frac{z}{c} = 0$  and  $1 + \frac{y}{b} = 0$  at the point then the line whose equations are (8) passes through the point.

The factors of (6) can be paired in another way. We write

$$\begin{aligned} & \left( \frac{x}{a} + \frac{z}{c} - \frac{y}{b} \right) \\ & t \left( \frac{x}{a} - \frac{z}{c} \right) = \left( 1 + \frac{y}{b} \right), \end{aligned}$$

and

$$\begin{cases} \frac{x}{a} - \frac{z}{c} = 0, \\ 1 - \frac{y}{b} = 0. \end{cases}$$

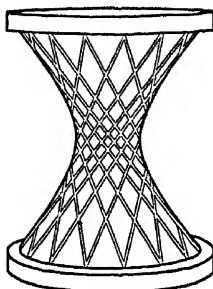


FIG. 152

These equations define a second family of rulings in every way analogous to the first and also lying on the hyperboloid, one ruling through every point of the surface.

Consider next the hyperbolic paraboloid whose equation is

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

This can be written

$$z = \left( \frac{x}{a} + \frac{y}{b} \right) \left( \frac{x}{a} - \frac{y}{b} \right).$$

We write the equations of two families of straight lines,

The first consists of lines whose equations are

$$= t \left( \frac{x}{a} + \frac{y}{b} \right),$$

$$t = \frac{x}{a} - \frac{y}{b}.$$

The second consists of lines whose equations are

$$\begin{cases} z = t \left( \frac{x}{a} - \frac{y}{b} \right) \\ t = \frac{x}{a} + \frac{y}{b}. \end{cases}$$

We have here a situation entirely analogous to the elliptic hyperboloid of one sheet. The student should go through the reasoning in detail.

### Exercises

Discuss the following surfaces. Sketch the surface, if possible, and if it has a center give its coordinates.

1.  $4x^2 + 25y^2 + 16z^2 = 100.$
2.  $4x^2 - 25y^2 + 16z^2 = 100.$
3.  $4x^2 - 25y^2 - 16z^2 = 100.$
4.  $z = 4x^2 - 8y^2.$
5.  $x^2 + y^2 = z^2.$
6.  $(2x + 3y - 4z)(6x - 4y + 2z - 1) = 0.$
7.  $(3x + 2y - z + 1)^2 = 0.$
8.  $x^2 + y^2 + 3z^2 + 1 = 0.$

Discuss in detail the rulings on the surfaces whose equations are:

9.  $\frac{x^2}{9} + \frac{y^2}{16} - \frac{z^2}{25} = 1.$

10.  $\frac{x^2}{9} - \frac{y^2}{16} + \frac{z^2}{25} = 1.$

11.  $x = \frac{z^2}{4} - \frac{y^2}{9}.$

## CHAPTER XV

### ELEMENTS OF CALCULUS — DERIVATIVES

**131. Functions.** We are already familiar with the notion of a variable. In mathematics the term **variable** simply means a letter which represents any one of a set of numbers, which set we sometimes call the domain or range of the variable.

If a variable  $y$  depends on another variable  $x$  for its value in such a way that when  $x$  is assigned a particular number of its range, the value of  $y$  is thereby determined, then  $y$  is said to be a **function**\* of  $x$ .

Consider the equation  $y = x^2$ ; here  $y$  is a function of  $x$ , since the value of  $y$  is determined when the value of  $x$  is given.

To say that  $y$  is a function of  $x$  usually means merely that  $y$  is equal to some expression in  $x$ , although the concept of function is much more general.

To express the fact that  $y$  is a function of  $x$ , we write  $y = f(x)$  (read " $y$  equals the  $f$ -function of  $x$ ," or simply " $y$  equals  $f$  of  $x$ "). The notation is very convenient when we are dealing with an unknown function of  $x$ , or when we wish to consider "any" function. Thus, the symbols  $f(0)$ ,  $f(1)$ ,  $f(a)$ ,  $f(a+b)$  mean the values of the function when  $x = 0, 1, a, a+b$ , respectively. For example, if  $f(x) = x^2 + 1$ ,  $f(0) = 1$ ,  $f(1) = 2$ ,  $f(a) = a^2 + 1$ ,  $f(a+b) = (a+b)^2 + 1$ .

In different problems,  $f(x)$  may represent different

\* More explicitly a single-valued function of  $x$ . If  $y$  has, in general, either of two values it is called a double-valued function, etc.

functions. However, if in the same problem several functions occur, we use different symbols to distinguish them, such as  $f(x)$ ,  $F(x)$ ,  $\phi(x)$ , etc.

In the equation  $y = f(x)$ , the variable  $x$  is called the **independent variable**, and the variable  $y$ , the **dependent variable**, and  $y$  is called an **explicit function** of  $x$ . If  $f(x, y) = 0$ , then  $y$  is an **implicit function** of  $x$  and  $x$  is an implicit function of  $y$ .

If  $x$  and  $y$  are thought of as rectangular coordinates of a variable point, a function  $y = f(x)$  usually can be represented by a curve whose equation is  $y = f(x)$ . This locus (curve) is called the **graph** of the function.

Conversely, whenever a locus is given by means of its equation in rectangular coordinates  $x, y$ , the variable  $y$  is thereby defined as a function of  $x$ . This function may be explicit, in which case the equation is solved for  $y$  in terms of  $x$ . Thus

$$y = mx + b,$$

$$y = 4px^2,$$

are examples of the equation of a straight line and of a parabola, where  $y$  is an **explicit function** of  $x$ . In such equations, on the other hand, as that of the circle

$$x^2 + y^2 = a^2,$$

$y$  is an **implicit function** of  $x$  and  $x$  is an **implicit function** of  $y$ . Here we can solve for  $y$  and obtain

$$y = \pm \sqrt{a^2 - x^2}.$$

In this case, to every value of  $x$  corresponds, in general, two values of  $y$ , namely,  $+\sqrt{a^2 - x^2}$  and  $-\sqrt{a^2 - x^2}$ ; that is,  $y$  is a double-valued function of  $x$ . To express

a function explicitly with respect to  $y$  or  $x$ , is by no means always a simple or even a possible matter. In other words, the equation defining the function may be so complicated as to make explicit representation out of the question.

### Exercises

1. If  $f(x) = 3x^2 - 2$  find  $f(0)$ ,  $f(1)$ ,  $f(-1)$ ,  $f(h)$ .
2. If  $f(x) = \frac{1}{x}$ , find  $f(1)$ ,  $f(4)$ ,  $f(\frac{1}{2})$ ,  $f(x^2 + 3x)$ ,  $f(-x)$ .
3. If  $F(x) = x^3 + 3x - 8$ , find  $F(0)$ ,  $F(-1)$ ,  $F(m)$ ,  $F(x-1)$ .
4. If  $Ax + By + C = 0$ , express  $y$  as an explicit function of  $x$ .
5. If  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , express  $y$  as an explicit function of  $x$ .
6. If  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , express  $y$  as an explicit function of  $x$ .
7. If  $f(x) = a^x$ , show that  $f(x) \cdot f(y) = f(x+y)$ .
8. If  $f(x) = \sin x$ , show that  $f(x+2\pi) = f(x)$ , and that  $f(-x) = -f(x)$ .
9. If  $f(x) = \log x$ , show that  $f(a) + f(b) = f(ab)$ .
10. If  $f(x) = \tan x$ , show that  $f(2x) = \frac{2f(x)}{1 - [f(x)]^2}$ .
11. If  $f(x) = \frac{x-1}{x+1}$ , show that  $\frac{f(a)-f(b)}{1+f(a)f(b)} = \frac{a-b}{1+ab}$ .
12. If  $f(x) = f(-x)$ , the function is called even. Mention four even functions.

13. If  $f(x) = -f(-x)$ , the function is called odd. Mention four odd functions.
14. Given  $x^2 + y^2 = 25$ . Is  $y$  an explicit or implicit function of  $x$ ?
15. Give three examples illustrating an explicit function; an implicit function.

**132. Increments — Limits.** When a variable changes from one numerical value to another, the difference found by subtracting the first value from the second is called the **increment** of the variable. If the variable is  $x$ , we shall denote an increment of  $x$  by  $\Delta x$  (read delta  $x$  and not delta times  $x$ ). Similarly,  $\Delta y$  will denote an increment of  $y$ ,  $\Delta f(x)$  an increment of  $f(x)$ , etc.

If  $P(x, y)$  is any point on the curve  $y = f(x)$ , then the coordinates of a neighboring point  $P'$  on the curve may be denoted by  $(x + \Delta x, y + \Delta y)$ .

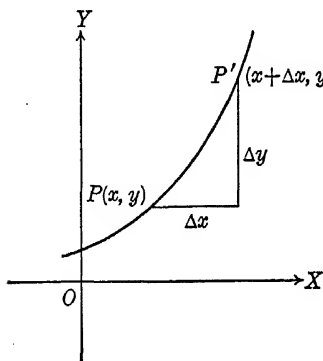


FIG. 153

The following four theorems concerning limits will be assumed without proof. The letters  $u$  and  $v$  stand for variables, the letter  $c$  for a constant.

**Theorem 1.** *The limit of  $u + v$  is the limit of  $u$  plus the limit of  $v$ .*

**Theorem 2.** *The limit of  $cv$  is  $c$  times the limit of  $v$ .*

**Theorem 3.** *The limit of  $uv$  is the limit of  $u$  times the limit of  $v$ .*

**Theorem 4.** *The limit of  $\frac{u}{v}$  is the limit of  $u$  divided by the limit of  $v$ , provided that the limit of  $v$  is not zero.*

*Example.* Find the limit of  $(u^2 + 3u)$  as  $u$  approaches the limit 1.

Solution: Symbolically we note

$$\lim_{u \rightarrow 1} (u^2 + 3u) = 4.$$

It is left as an exercise for the student to explain the applications of Theorems 1 and 2.

### Exercises

Find the limit, when it exists, of the following expressions. In each case state which theorems on limits you used.

1.  $\lim_{x \rightarrow 1} (x^2 + 3x - 1).$
2.  $\lim_{x \rightarrow 0} \frac{x^2 + 2x}{x}.$
3.  $\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x^2 - 1}.$
4.  $\lim_{\Delta x \rightarrow 0} [2 + 3\Delta x - 50(\Delta x)^2].$
5.  $\lim_{\Delta x \rightarrow 0} \frac{4\Delta x - 7(\Delta x)^2}{\Delta x}.$
6.  $\lim_{\Delta x \rightarrow 0} \frac{(2 - \Delta x)^2 - 4}{\Delta x}.$
7.  $\lim_{\Delta x \rightarrow 0} \frac{\frac{1}{a + b\Delta x} - \frac{1}{a}}{\Delta x}.$
8.  $\lim_{\Delta x \rightarrow 0} \frac{\frac{4}{4 + x} - \frac{4}{x}}{\Delta x}.$

**133. The derivative.** We now return to our problem of determining the slope of the tangent to a curve, and will formulate it more generally.

We assume that the equation of the curve is given in

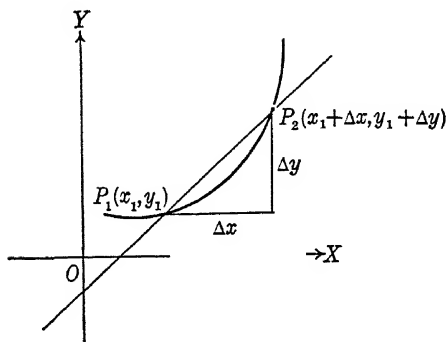


FIG. 154

the form  $y = f(x)$ . We take a point  $P_1(x_1, y_1)$  on the curve, so that we have

$$y_1 = f(x_1)$$

and we take a neighboring point  $P_2(x_1 + \Delta x, y_1 + \Delta y)$  on the curve, so that we have

$$y_1 + \Delta y = f(x_1 + \Delta x).$$

$$\therefore \Delta y = f(x_1 + \Delta x) - f(x_1).$$

Then,

$$\text{slope of } P_1P_2 = \frac{\Delta y}{\Delta x}$$

and the slope of the tangent at  $P_1$  is the limit approached by  $\frac{\Delta y}{\Delta x}$ , as  $\Delta x$  approaches 0. In symbols

$$\begin{aligned}\text{slope of tangent at } P_1 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}, \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}\end{aligned}$$

The limit (when it exists), of the ratio  $\frac{\Delta y}{\Delta x}$  as  $\Delta x$  approaches zero, is called the derivative of  $y$  with respect to  $x$ . The derivative is designated by the symbol  $D_x y$ :

$$D_x y = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Sometimes the symbols  $\frac{dy}{dx}$ ,  $y'$ ,  $f'(x)$  are used.

*Example 1.* Find the derivative of  $y = x^3 - 2x + 1$ .

*Solution:*

1.  $y + \Delta y = (x + \Delta x)^3 - 2(x + \Delta x) + 1$
- $y + \Delta y = x^3 + 3x^2 \Delta x + 3x \overline{\Delta x}^2 + \overline{\Delta x}^3 - 2x - 2\Delta x + 1.$
2.  $y = x^3 - 2x + 1.$
3.  $\Delta y = 3x^2 \Delta x + 3x \overline{\Delta x}^2 + \overline{\Delta x}^3 - 2\Delta x$
- $\frac{\Delta y}{\Delta x} = 3x^2 + 3x \overline{\Delta x} + \overline{\Delta x}^2 - 2.$
4.  $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = D_x y = 3x^2 - 2.$

*Example 2.* Find the derivative of  $y^2 = x$ .

*Solution 1.*  $(y + \Delta y)^2 = x + \Delta x.$

$$y^2 + 2y \Delta y + \overline{\Delta y}^2 = x + \Delta x.$$

$$2. \quad y^2 = x.$$

$$3. \quad 2y \Delta y + \overline{\Delta y}^2 = \Delta x.$$

$$\Delta y (2y + \Delta y) = \Delta x.$$

$$\Delta y = \frac{\Delta x}{2y + \Delta y}.$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{2y + \Delta y}.$$

$$4. \quad \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = D_x y = \frac{1}{2y}.$$

*Example 3.* Find the derivative of  $y = \frac{2}{x}$ .

Solution 1.  $y + \Delta y = \frac{2}{x + \Delta x},$

$$2. \quad y = \frac{2}{x},$$

$$3. \quad \Delta y = \frac{2}{x + \Delta x} - \frac{2}{x} = \frac{-2\Delta x}{x(x + \Delta x)}$$

$$\frac{\Delta y}{\Delta x} = \frac{-2}{x(x + \Delta x)},$$

$$4. \quad \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = D_x y = -\frac{2}{x^2}.$$

### Exercises

Find the derivative of  $y$  in each of the following examples.

1.  $y = 2x.$

2.  $y = x^2 + x.$

3.  $y = x + 5.$

4.  $y = x^3.$

5.  $y = x^3 - 3x.$

6.  $y = \frac{2}{x}.$

7.  $y = \frac{1}{x^2}$ .      8.  $y = x + \frac{1}{x}$ .      9.  $y^2 = x$ .
10.  $x^2 + y^2 = 5$ .      11.  $y^2 = 3x + 1$ .      12.  $2x^2 - y^2 = 2$ .
13.  $y = x$ .      14.  $y = 3 - x$ .      15.  $y = x^2 + \frac{1}{x}$ .

**134. General theorems on derivatives.** To determine the derivative of a function from the definition would, in the case of the more complicated functions, be not only very tedious, but often very difficult. Fortunately, a few simple general theorems or formulas enable us to avoid such long computations and difficult limit evaluations. We proceed to derive some of these theorems.

I. *The derivative of a constant is zero.* In symbols, if  $y = c$ , where  $c$  is a constant,

$$D_x c = 0.$$

Proof  $y + \Delta y = c$ .

$$y = c.$$

$$\Delta y = 0.$$

$$\frac{\Delta y}{\Delta x} = 0.$$

$$D_x y = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = 0,$$

or  $D_x y = D_x c = 0$ .

II. *The derivative of a constant times any function is the constant times the derivative of the function.* In

symbols, if  $y = cu$ , where  $u$  is any function of  $x$  and  $c$  is a constant,

$$D_x cu = cD_x u.$$

Proof  $y + \Delta y = c(u + \Delta u).$

$$y = cu.$$

$$\Delta y = c \Delta u.$$

$$\frac{\Delta y}{\Delta x} = c \frac{\Delta u}{\Delta x}.$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = c \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x}.$$

or,  $D_x y = D_x cu = cD_x u.$

III. *The derivative of a sum of functions is equal to the sum of the derivatives of the functions.* In symbols, if

$$y = u + v + \dots + w,$$

where  $u, v, \dots, w$  are functions of  $x$ ,

$$D_x(u + v + \dots + w) = D_x u + D_x v + \dots + D_x w.$$

Proof: We have

$$y + \Delta y = (u + \Delta u) + (v + \Delta v) + \dots + (w + \Delta w).$$

$$y = u + v + \dots + w.$$

$$\Delta y = \Delta u + \Delta v + \dots + \Delta w.$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} + \dots + \frac{\Delta w}{\Delta x}.$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} + \dots + \lim_{\Delta x \rightarrow 0} \frac{\Delta w}{\Delta x}$$

or 
$$D_x y = D_x u + D_x v + \dots + D_x w.$$

IV. *The derivative of  $x^n$  is  $nx^{n-1}$ .* This is true for any constant exponent  $n$ . We shall prove the theorem at this point on the assumption that  $n$  is a positive integer (whole number).

Proof: 
$$y + \Delta y = (x + \Delta x)^n.$$

Expanding by the binomial theorem, we have

$$y + \Delta y = x^n + nx^{n-1} \Delta x + \frac{n(n-1)}{2} x^{n-2} \overline{\Delta x^2} + \dots + \overline{\Delta x^n}.$$

$$y = x^n.$$

$$\Delta y = nx^{n-1} \Delta x + \frac{n(n-1)}{2} x^{n-2} \overline{\Delta x^2} + \dots + \overline{\Delta x^n}.$$

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{2} x^{n-2} \Delta x + \text{terms involving}$$

higher powers of  $\Delta x$ .

$$= nx^{n-1} + \Delta x \cdot E,$$

where  $E$  is the expression which remains after  $\Delta x$  has been taken out as a factor from all terms after the first. The limit approached by the right-hand member of this relation, when  $\Delta x \rightarrow 0$ , is  $nx^{n-1}$ .

$$\therefore D_x y = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = nx^{n-1}.$$

*Example.* Given  $y = x^3 + 3x^2 - 2x + 1$ . Find  $D_x y$  and the slope of the curve at the point  $(1, 3)$

Solution:

$$\begin{aligned} D_x y &= D_x x^3 + D_x 3x^2 + D_x (-2x) + D_x 1, \text{ by III,} \\ &= D_x x^3 + 3 \cdot D_x x^2 - 2 D_x x + D_x 1, \quad \text{by II,} \\ &= 3x^2 + 6x - 2, \quad \text{by IV and I.} \end{aligned}$$

The slope of the curve at  $(1, 3)$  is  $3(1)^2 + 6(1) - 2 = 7$ .

We have here separated the steps so as to indicate the successive application of theorems I to IV. The student will find no difficulty in writing down the final result immediately from the given polynomial.

### Exercises

Write the derivatives of each of the following functions. Find the value of the slope of the tangent to the corresponding curve at the point indicated.

1.  $y = x^3 - 4x$ ;  $(2, 0)$ .
2.  $y = x^2 - 2x + 3$ ;  $(3, 6)$ .
3.  $y = x^3 - 3x^2 + 6x - 1$ ;  $(1, 3)$ .
4.  $y = x^4 - 4x^2 + 1$ ;  $(2, 1)$ .
5.  $y = x^5 - 10x^2 + 3$ ;  $(1, -6)$ .
6. Find the values of  $x$  for which the slope of the tangent to the curve  $y = x^3 - 3x$  is zero. What is the slope of the tangent at the origin? Locate the corresponding points and use the information obtained to sketch the curve.
7. Find the values of  $x$  for which the tangents to the curve  $y = \frac{1}{4}x^4 - 2x^2 + 1$  are parallel to the  $x$ -axis.
8. Find the equations of the tangent and the normal to the curves of Exs. 1-5 at the points indicated.

**135. The derivative of a product.** Let  $y = uv$ , where  $u$  and  $v$  are two functions of  $x$  which have derivatives. Let the changes in  $u$ ,  $v$ , and  $y$  due to a change  $\Delta x$  in  $x$  be  $\Delta u$ ,  $\Delta v$ , and  $\Delta y$ , respectively. We then have

$$\begin{aligned} y + \Delta y &= (u + \Delta u)(v + \Delta v), \\ &= uv + u \Delta v + v \Delta u + \Delta u \cdot \Delta v. \end{aligned}$$

$$y = uv.$$

$$\Delta y = u \Delta v + v \Delta u + \Delta u \cdot \Delta v.$$

and 
$$\frac{\Delta y}{\Delta x} = u \cdot \frac{\Delta v}{\Delta x} + v \cdot \frac{\Delta u}{\Delta x} + \Delta u \cdot \frac{\Delta v}{\Delta x}.$$

In order to find the derivative  $y'$  of  $y$ , we must find the limit of both members of the equation as  $\Delta x \rightarrow 0$ , i.e.,

$$\begin{aligned} y' = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= u \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} + v \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} + \\ &\quad \lim_{\Delta x \rightarrow 0} \Delta u \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x}. \end{aligned}$$

Of the limits in the right-hand side of this equation, the first and last are by definition, the derivative,  $v'$ , of  $v$ ; the second is equal to the derivative,  $u'$ , of  $u$ . Also in the last term  $\Delta u$  approaches 0, when  $\Delta x \rightarrow 0$ . Hence we have

$$y' = uv' + vu',$$

or 
$$D_x(uv) = uD_x v + vD_x u.$$

It is well to remember this formula in words:

**The derivative of the product of two functions is equal**

to the first function times the derivative of the second plus the second times the derivative of the first.

*Example.* Find  $D_x y$  given  $y = (x + 1)(x^2 - 1)$ .

Solution:  $D_x y = (x + 1) \cdot 2x + (x^2 - 1) \cdot 1$ ,

$$= 2x^2 + 2x + x^2 - 1,$$

$$= 3x^2 + 2x - 1.$$

This is the same result as that obtained by multiplying out the product and then finding the derivative, that is, by finding the derivative of  $y = x^3 + x^2 - x - 1$ .

**136. The derivative of a quotient.** Let  $y = \frac{u}{v}$  where

$u$  and  $v$  are functions of  $x$ , which have derivatives, and let  $\Delta x$ ,  $\Delta u$ ,  $\Delta v$ , and  $\Delta y$  have the same significance as in the last article. We then have

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$$

$$y = \frac{u}{v}.$$

$$\begin{aligned} \Delta y &= \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}, \\ &= \frac{v \Delta u - u \Delta v}{v(v + \Delta v)}. \end{aligned}$$

$$\text{and} \quad \frac{\Delta y}{\Delta x} = \frac{v \frac{\Delta u}{\Delta x} - u \frac{\Delta v}{\Delta x}}{v(v + \Delta v)}.$$

Therefore, taking the limit of both sides of the equation

as  $\Delta x \rightarrow 0$ , and remembering that as  $\Delta x$  approaches 0,  $\Delta v$  also approaches 0, we have

$$D_x \frac{u}{v} = \frac{v D_x u - u D_x v}{v^2}.$$

This formula also should be remembered in words:

The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator all divided by the square of the denominator.

*Example.* Find  $D_x y$  given  $y = \frac{x}{x^2 + 1}$

Solution: In applying the formula it is good practice to begin by writing the form of a fraction, with the square of the given denominator in the denominator and a blank numerator, thus,

$$D_x y = \frac{\quad}{(x^2 + 1)^2},$$

and then fill in the numerator:

$$D_x y = \frac{(x^2 + 1) \cdot (1) - x \cdot (2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

If the numerator of the fraction is constant, *i.e.*,  $u = c$ ,  $D_x u = 0$ , and the formula becomes

$$D_x \frac{c}{v} = -\frac{c}{v^2} D_x v.$$

For example, if  $y = \frac{1}{x}$ ,  $y' = -\frac{1}{x^2}$ ;

$$\text{if } y = \frac{1}{x^2}, \quad y' = -\frac{2x}{x^4} = -\frac{2}{x^3}.$$

## Exercises

Find the derivative of each of the following functions:

1.  $y = (x^2 + 1)(2x - 1)$ .      2.  $y = (x^3 - 1)(x^2 + 1)$ .

3.  $y = (x^2 + x + 1)(x - 1)$ .

4.  $y = (x^3 - 2x^2 + 4x - 2)(x^2 + 1)$ .

5.  $y = \frac{x^2 + 1}{x}$ .

6.  $y = \frac{x - 1}{x + 1}$ .

7.  $y = \frac{x - 1}{x^2 + 1}$ .

8.  $y = \frac{1}{x^3}$ .

9. By the product formula verify that the derivative of  $y = x^2 = x \cdot x$  is  $2x$ . Assuming the latter, prove that if  $y = x^3 = x^2 \cdot x$ ,  $y' = 3x^2$ . Could this method be used to derive the formula  $D_x x^n = nx^{n-1}$ , when  $n$  is a positive integer?

10. Show that the formula  $D_x cu = cD_x u$  is a special case of the general product formula.

11. Show that  $D_x \frac{1}{x^k} = -\frac{k}{x^{k+1}}$ , where  $k$  is a positive integer.

Hence show that  $D_x x^n = nx^{n-1}$ , when  $n$  is a negative integer.

**137.** The derivative of  $u^n$ . If  $u$  is a function of  $x$ , which has a derivative, what is the derivative of  $y = u^n$ ? If we denote by  $\Delta u$  and  $\Delta y$ , the changes in  $u$  and  $y$  respectively, brought about by a change  $\Delta x$  in  $x$ , we have at once the identity

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}, \text{ and}$$

$$D_x y = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x}.$$

Since  $\Delta u \rightarrow 0$ , when  $\Delta x \rightarrow 0$ , we can write the first factor on the right

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} = \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u}.$$

The last limit is, however, by definition, the derivative of  $y = u^n$ , where  $u$  is considered as the independent variable, and we know that this derivative is  $nu^{n-1}$  ( $n$  a positive integer). Hence, we have

$$D_x u^n = nu^{n-1} D_x u.$$

*Example.* Find the derivative of  $(x^2 + 1)^5$ .

Solution: We could expand this expression by the binomial theorem and then find the derivative of the resulting polynomial. But the above formula enables us to write down the desired result without this tedious work. In this case  $u = x^2 + 1$ ,  $D_x u = 2x$ , and, therefore,

$$D_x (x^2 + 1)^5 = 5 (x^2 + 1)^4 \cdot 2x = 10x (x^2 + 1)^4.$$

**138. Implicit functions.** One of the most useful applications of the last formula is to be found in the fact that it enables us to find the derivative of a function when this function is defined by an implicit relation between  $x$  and  $y$ .

*Example 1.* Find  $D_x y$  given  $x^2 + y^2 = a^2$ .

Solution: If we take the derivative of both sides of the equation with respect to  $x$ , we get

$$D_x (x^2 + y^2) = D_x a^2,$$

$$\text{or} \quad D_x x^2 + D_x y^2 = D_x a^2,$$

$$\text{or} \quad 2x + 2y D_x y = 0.$$

We can solve this equation for  $D_x y$ , and obtain

$$D_x y = -\frac{x}{y},$$

a result with which we are already familiar. (See § 48.)

*Example 2.* Find  $D_x y$  given  $x^3 + y^3 - 3xy = 0$ .

Solution: By taking the derivative of each term with respect to  $x$  we get

$$3x^2 + 3y^2 D_x y - 3(x D_x y + y \cdot 1) = 0.$$

Solving this equation for  $D_x y$ , we obtain

$$D_x y = -\frac{x^2 - y}{y^2 - x}.$$

The slope of the tangent at any point  $(x_1, y_1)$  on the curve, for which  $y_1^2 - x_1 \neq 0$ , is then

$$-\frac{x_1^2 - y_1}{y_1^2 - x_1}.$$

**139.** The derivative of  $x^n$  when  $n$  is any rational constant. Let  $n = \frac{p}{q}$  where  $p, q$  are positive integers;

then  $y = x^n = x^{\frac{p}{q}}$ . If we raise both sides to the  $q$ th power, we have

$$y^q = x^p$$

and, by taking the derivative of both sides,

$$qy^{q-1} D_x y = px^{p-1},$$

$$\begin{aligned}\text{or } D_x y &= \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{x^{\frac{p}{q}(q-1)}} = \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-\frac{p}{q}}} \\ &= \frac{p}{q} \cdot x^{\frac{p}{q}-1} = nx^{n-1}.\end{aligned}$$

which proves that the formula  $D_x x^n = nx^{n-1}$  holds also when  $n$  is any positive rational constant. If  $n$  is any negative rational constant,  $n = -k$ , we have

$$y = x^n = x^{-k} = \frac{1}{x^k}$$

$$\text{and } D_x y = -\frac{kx^{k-1}}{x^{2k}} = -k \cdot x^{-k-1} = nx^{n-1}.$$

The formula is then valid also when  $n$  is negative:

For any rational constant  $n$ ,  $D_x x^n = nx^{n-1}$ . Also, more generally, for any function  $u$  and any rational constant  $n$ ,  $D_x u^n = nu^{n-1} D_x u$ .

For example:  $\sqrt{x^2 + 4}$  can be written  $(x^2 + 4)^{\frac{1}{2}}$ . Hence,

$$D_x \sqrt{x^2 + 4} = D_x (x^2 + 4)^{\frac{1}{2}} = \frac{1}{2} (x^2 + 4)^{-\frac{1}{2}} 2x = x$$

### Exercises

Find the derivative of  $y$  in each of the following cases:

1.  $y = (2x + 1)^2$ .
2.  $y = (x^2 - 1)^3$ .
3.  $y = (x^3 - 1)^{21}$ .
4.  $y = (3x^2 - 2x - 7)^5$ .
5.  $y = (7x^2 - 8x - 2)^{25}$ .
6.  $y^2 = 4x$ .
7.  $4x^2 + y^2 = 4$ .
8.  $x^2 - 4y^2 = 4$ .

$$9. x^3 - y^3 + 2xy = 0. \quad 10. x^2 + y^2 - 2xy - 7x = 0.$$

$$11. x^2 + y^2 - 3xy - 2x - 8y - 7 = 0.$$

$$12. y = x^{\frac{1}{3}}. \quad 13. y = \sqrt[3]{x^2}.$$

$$14. y = \sqrt{x^2 + 7}. \quad 15. y = \sqrt[3]{x^2 - 6x + 8}.$$

$$16. y = \sqrt[4]{3x^2 - 6x - 2}. \quad 17. y = \sqrt[4]{\frac{x}{x^3 - 5}}$$

$$18. y = (2x + 1)\sqrt{x^2 + 3}.$$

$$19. y = \sqrt{x^2 + 5} \sqrt[3]{7x + 3}.$$

$$20. y^z = \frac{6}{5x - 2} \quad 21. y = \sqrt[3]{\frac{4x - 1}{3x^2 + 2}}.$$

22. Prove, by finding the derivative of  $y$ , that the slope of the tangent to the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  at the point  $(x_1, y_1)$  is  $-\frac{b^2x_1}{a^2y_1}$ .

23. Derive results for the hyperbola  $b^2x^2 - a^2y^2 = a^2b^2$  analogous to those found in Ex. 22.

24. Prove that the slope of the tangent to the conic  $Ax^2 + By^2 + Dx + Ey + C = 0$  at the point  $(x_1, y_1)$  is

$$-\frac{2Ax_1 + D}{2By_1 + E}.$$

25. The derivative of the first derivative is called the second derivative and is denoted by the symbol  $D_x^2y$ . Find  $D_x^2y$  in Ex. 1-12.

## CHAPTER XVI

### APPLICATIONS OF THE DERIVATIVE

**140. Tangents and normals.** If the equation of a curve is given in the form  $y = f(x)$ , the slope  $m$  of the tangent at the point  $P_1(x_1, y_1)$  on the curve is the value of the derivative  $f'(x)$ , when  $x = x_1$ , i.e.,  $m = f'(x_1)$ . Hence, the equation of the tangent at  $P_1$  is

$$y - y_1 = f'(x_1) (x - x_1),$$

and the equation of the normal at  $P_1$  is

$$y - y_1 = -\frac{1}{f'(x_1)} (x - x_1).$$

### Exercises

Find the equations of the tangent and the normal for each of the following curves at the points indicated:

1.  $y = x^3 - 3x$  at  $(0, 0)$ ; at  $(2, 2)$ .

2.  $y^2 = x^3$  at  $(1, 1)$ ; at  $(4, 8)$ .

3.  $y = x^3$  at  $(1, 1)$ .

4.  $y = \frac{1}{4}x^4$  at  $(2, 4)$ .

5.  $x^2 + y^2 - 4x + 4y + 6 = 0$  at  $(1, -1)$ .

6.  $x^2 + 4y^2 - 4x + 16y - 17 = 0$  at  $(1, 1)$ .

7. Prove that the equation of the tangent to the curve

$Ax^2 + By^2 + Dx + Ey + C = 0$  at the point  $(x_1, y_1)$  on the curve, is

$$Ax_1x + By_1y + \frac{1}{2}D(x + x_1) + \frac{1}{2}E(y + y_1) + C = 0.$$

**141. Increasing and decreasing functions.** If a curve is defined by the equation  $y = f(x)$ , where  $f(x)$

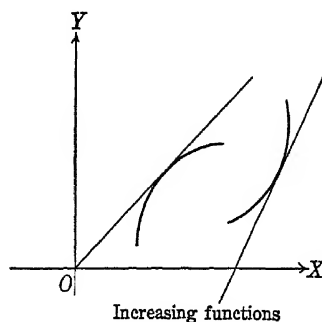


FIG. 155

is single-valued, continuous and differentiable, the derivative  $D_x y$  furnishes a simple method of determining whether the curve is rising or falling, that is, whether

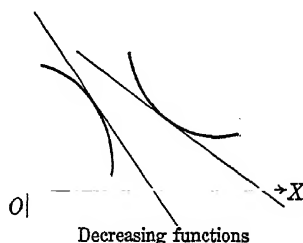


FIG. 156

the function is increasing or decreasing. It will be noted, reading from left to right, that  $x$  increases, and

that when the curve rises the slope is positive and that when it falls the slope is negative. Therefore, when

$D_x y$  is +,  $y$  increases;

$D_x y$  is -,  $y$  decreases.

*Example.* For what values of  $x$  is  $y = x^3 - 27x$  increasing? decreasing?

Solution:

$$D_x y = 3x^2 - 27 = 3(x - 3)(x + 3).$$

The slope is positive for values of  $x < -3$ , negative in the interval  $-3 < x < 3$  and positive again when  $x > 3$ . Therefore, the curve rises until  $x = -3$ , falls in the interval from  $x = -3$  to  $x = 3$  and rises thereafter.

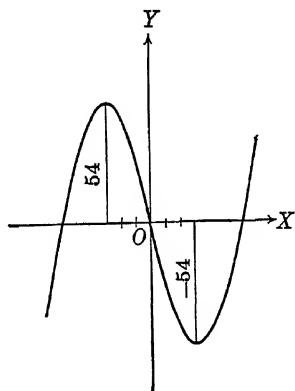


FIG. 157

### Exercises

For what values of  $x$  are the following functions increasing? decreasing?

1.  $y = 7x - 3.$

2.  $y = x^2 + 5.$

3.  $y = x^3 - 27x - 1.$

4.  $y = \frac{x^3}{3} - x^2 + 1.$

5.  $y = x^3 + 2x^2 - 15x - 1.$

6.  $y = \frac{1}{2}x^4 - x^3 - x^2 + 7.$

**142. Turning points.** Maximum and minimum points. A point at which a curve stops rising and be-

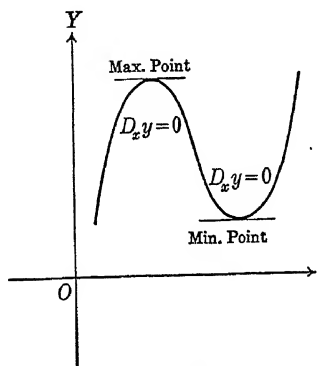


FIG. 158

gins to fall is called a **maximum** point; a point at which it stops falling and begins to rise, is called a **minimum** point. In either case the point is called a **turning point**. If  $D_x y$  is continuous,  $D_x y$  cannot change from positive

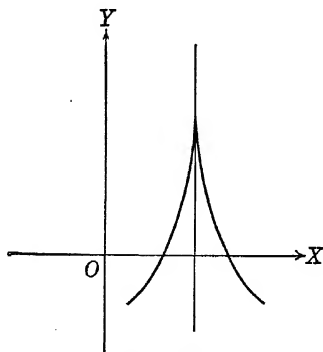


FIG. 159

to negative or *vice versa* without passing through the value 0. Hence the abscissas of the turning points are obtained from the equation  $D_x y = 0$ . The tangents at the turning points are horizontal.

A curve may also cease rising and begin to fall (or *vice versa*) at a point where the tangent is vertical, as shown in Fig. 159. Here also the slope of the tangent changes from positive to negative (or *vice versa*), i.e.,  $D_x y$  changes sign. But  $D_x y$  is not continuous at such a point.

### Exercises

For what values of  $x$  are the following functions increasing? decreasing? Find the turning points.

1.  $y = 6x - 1$ .

2.  $y = x^2 + 5$ .

3.  $y = x^2 + 4x + 5$ .

4.  $y = x^3 - 27x + 6$ .

5.  $y = x^3 + 2x^2 - 15x + 7$ .

6.  $y = x^4 - 2x^3 - 2x^2 - 7$ .

7.  $y = \frac{2x}{x-1}$ .

8.  $y = x - 2$ .

**143. Concavity.** A curve is said to be **concave upward** when its slope is increasing and **concave downward** when its slope is decreasing.

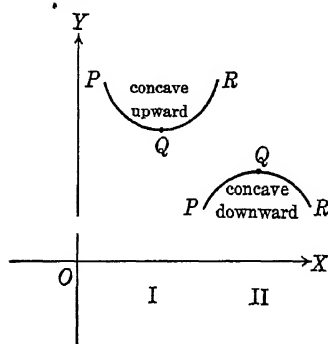


FIG. 160

ward when its slope is decreasing. Let us consider curve I in Fig. 160. The slope at  $P$  is  $-$ , at  $Q$  it is  $0$ , at  $R$  it is  $+$ . Therefore, the slope is increasing and the curve is concave upward. In Fig. II, the slope at  $P$  is  $+$ , at  $Q$  it is  $0$ , at  $R$  it is  $-$ . Therefore, the slope is decreasing and the curve is concave downward.

Hence, a curve is

Concave upward when  $D_x^2y > 0$ ,

Concave downward when  $D_x^2y < 0$ .

**144. Points of inflection.** A point at which a curve changes from concave upward to concave downward or *vice versa*, is called a point of inflection.

Such a point is a point at which  $D_x^2y$  changes sign, either from positive to negative, or from negative to positive. If  $D_x^2y$  is a continuous function of  $x$ , this can

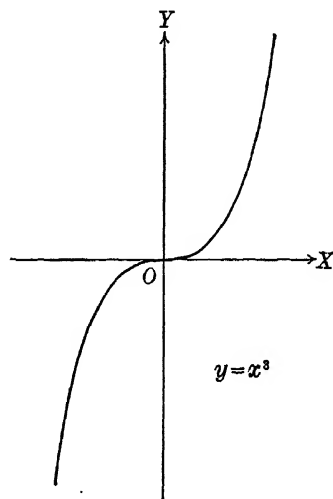


FIG. 161

happen only when it passes through 0. Hence, when  $D_x^2 y$  is continuous at a point of inflection,  $D_x^2 y = 0$ .

The condition  $D_x^2 y = 0$  is necessary but not sufficient to assure us that the point is a point of inflection. It is sufficient to know that  $D_x^2 y$  changes sign at the point; that is, if  $D_x^2 y$  changes sign there is a point of inflection. For example, consider the curve  $y = x^3$ . We have  $y' = 3x^2$ ,  $y'' = 6x$ . We note that  $y'' = 0$ , when  $x = 0$ , and that  $y''$  is positive for all positive values of  $x$  and negative for all negative values of  $x$ ;  $y''$  does then change sign as  $x$  passes through 0. The curve is concave downward to the left of the  $y$ -axis and concave upward to the right; it has a point of inflection at the origin.

On the other hand, the curve  $y = x^4$ , gives  $y' = 4x^3$ ,  $y'' = 12x^2$ . We again have  $y'' = 0$ , when  $x = 0$ , but  $y''$  is positive for all other values of  $x$ . The curve is concave upward throughout; it does not have a point

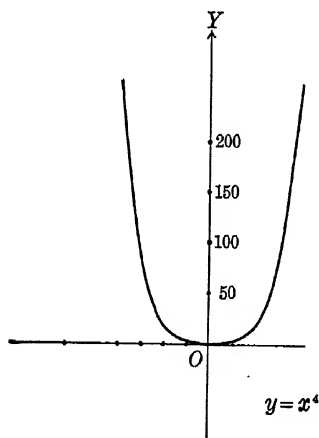


FIG. 162

of inflection at the origin, even though at that point  $y'' = 0$ .

**145. Test for maxima or minima.** The tangent is horizontal at any point for which  $D_x y = 0$ . Such a point is either a maximum point, a minimum point or a point of inflection with a horizontal tangent. If the point is a maximum, the curve is concave downward; if the point is a minimum, the curve is concave upward. Therefore, we have the test:

$D_x y = 0$ ,  $D_x^2 y$  negative, a maximum point;

$D_x y = 0$ ,  $D_x^2 y$  positive, a minimum point.

A point for which  $D_x y = 0$  and  $D_x^2 y = 0$ , is usually a point of inflection, but in exceptional cases it may be a maximum or minimum point.

*Example.* Trace the curve whose equation is

$$y = x^3 - 3x.$$

Solution: We note first that

$$D_x y = 3x^2 - 3,$$

$$D_x^2 y = 6x.$$

By placing  $D_x y = 0$  we obtain the so-called critical values of  $x$ , i.e., the abscissas of the points at which the tangent to the curve is horizontal. Thus

$$3(x^2 - 1) = 0$$

or

$$x = \pm 1.$$

The corresponding values of  $y$  give the critical points

$$(1, -2) \text{ and } (-1, 2).$$

Moreover, since  $x = 1$  makes  $D_{xy}^2$  positive, the point  $(1, -2)$  is a minimum point; since  $x = -1$  makes  $D_{xy}^2$  negative, the point  $(-1, 2)$  is a maximum point.

By placing  $D_x^2y = 0$  we find  $x = 0$  and the corresponding value of  $y$  is 0. Since  $D_x^2y$  is negative to the left of the origin and positive to the right, this point is a point of inflection and the curve is concave downward at every point to the left of the origin and concave upward at every point to the right of the origin. The slope of the inflectional tangent is  $-3$  and the equation of the inflectional tangent is  $y = -3x$ .

The results obtained may be tabulated as follows:

$x$	$y$	$D_x y$	$D_{xy}^2$	
1	-2	0	+	Min.
-1	2	0	-	Max.
0	0	-3	0	Infl.

We shall now plot these three points and draw the tangents at them.

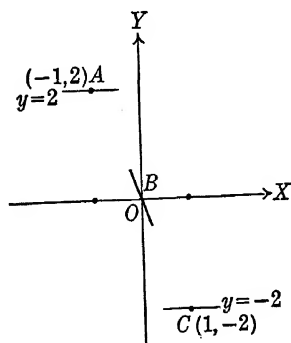


FIG. 163

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Since  $A$  is a maximum point the curve at that point has the appearance of the curve  $A$  given in Fig. 164. At

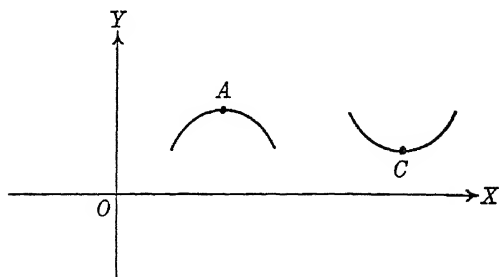


FIG. 164

the minimum point  $C$ , the curve has the appearance of the curve  $C$  given in Fig. 164. At the left of the point  $B$

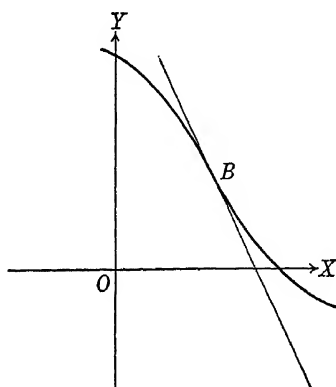


FIG. 165

the curve is concave downward while at the right it is concave upward. The curve, therefore, has the appearance of the curve given in Fig. 165.

It is now possible to draw the curve if we keep in

mind the general appearance at each critical point. However, we must still determine what the curve looks like at the left of  $A$  and the right of  $C$ . To this end we will plot the two points whose abscissas are  $x = -2$  and  $x = 2$ . Our table now becomes

$x$	$y$	$D_x y$	$D_x^2 y$	
1	-2	0	+	Min.
-1	2	0	-	Max.
0	0	-3	0	Infl.
-2	-2	9	-12	Concave downward.
2	2	9	12	Concave upward.

The curve is

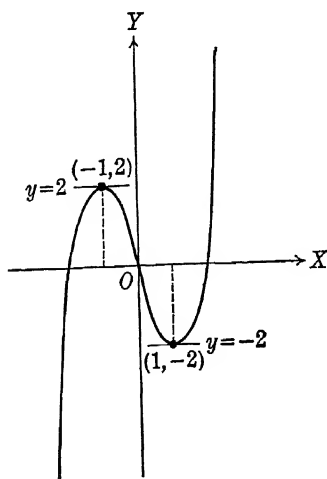


FIG. 166

## Exercises

Trace the following curves. In each case find the maximum and minimum points, the points of inflection and draw the horizontal tangents and the inflectional tangents.

1.  $y = \frac{x^3}{3} + \frac{x^2}{2} - 2x + 1.$

2.  $y = \frac{x^3}{3} - \frac{5x^2}{2} + 6x + 1.$

3.  $y = x^3 + 3x^2 + 1.$

4.  $y = x^3 - 3x^2 + 1.$

5.  $y = 7 - 3x^2 - x^3.$

6.  $y = 4 + 2x + x^2 - x^3.$

7.  $y = x^3 - 2x^2 + x.$

8.  $y = \frac{x}{3} - x^2 + 1.$

9.  $y = 3 + 15x - 2x^2 - x^3.$

10.  $y = 2x^3 - \frac{5}{2}x^2 - 3x + 1.$

**146. Maxima and minima problems.** The theory of maxima and minima can be applied to many practical problems. Some of these problems are quite involved, but many can be solved with the small amount of calculus we have studied. The following examples will illustrate the methods.

*Example 1.* From a square sheet of cardboard 12 inches on a side, equal squares are cut from the corners and the resulting projections turned up to form a rectangular box. What should be the dimensions of the squares cut from the corners in order that the resulting box may have the maximum capacity?

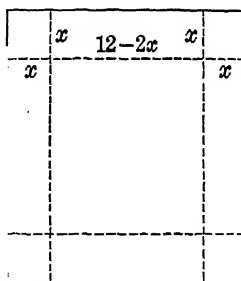


FIG. 167

**Solution:** Let  $x$  be the length of the side of the square

to be cut out. The resulting box will have a square base  $12 - 2x$  inches on a side; its depth will be  $x$  inches. Its volume,  $V$ , which we wish to make a maximum, will then be, in cubic inches,

$$\begin{aligned} V &= (12 - 2x)^2 x = 4x(6 - x)^2 \\ &= 4(36x - 12x^2 + x^3). \end{aligned}$$

Looked at as an algebraic formula,  $x$  might have any value, but from the nature of our problem  $x$  cannot be negative nor can it be greater than 6.

$$\begin{aligned} V' &= 4(36 - 24x + 3x^2) \\ &= 12(12 - 8x + x^2) \\ &= 12(x - 6)(x - 2). \\ V'' &= 12(-8 + 2x). \end{aligned}$$

The critical values of  $x$  are given by placing  $V' = 0$ ; this gives  $x = 6$  or  $x = 2$ . The value  $x = 6$  makes  $V'' = +48$  and so is not the value we seek. The value  $x = 2$  makes  $V'' = -48$  and hence  $x = 2$  must give the desired maximum. The dimensions of the maximum box are therefore 8 in. by 8 in. by 2 in.; the maximum capacity is 128 cu. in.

*Example 2.* A closed cylindrical tin can of given volume is to be constructed. What should be the ratio between the diameter of the base and the height of the can, if the amount of tin to be used is a minimum?

Solution: Let  $S$  = the total surface or amount of tin to be used and  $x$  and  $y$

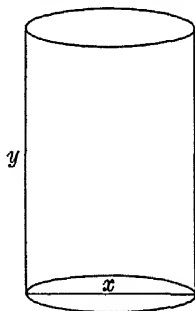


FIG. 168

the diameter and altitude of the cylindrical can. Now

$$\begin{aligned} S &= 2 \cdot \frac{\pi x^2}{4} + \pi xy, \\ (1) \quad &= \pi \left( \frac{x^2}{2} + xy \right). \end{aligned}$$

Since the volume is a constant the variables  $x$  and  $y$  are connected by the relation,

$$\frac{\pi x^2}{4} y = V,$$

$$(2) \text{ or } x^2 y = \frac{4V}{\pi} = k^3 \text{ where } k^3 \text{ is a constant.}$$

From this point on there are two methods of procedure.

*Method I.* By means of (2) we can eliminate one of the variables, say  $y$ , and obtain  $S$  as a function of  $x$  alone:

$$S = \pi \left( \frac{x^2}{2} + \frac{k^3}{x} \right).$$

$$\text{Now} \quad S' = \pi \left( x - \frac{k^3}{x^2} \right).$$

Since  $S$  is to be a minimum,  $S' = 0$ , which gives  $x = k$  as the only critical value. The corresponding value of  $y$  is  $y = \frac{k^3}{x^2} = k$ . The minimum amount of tin is then used when  $y = x$ , i.e., when the height is equal to the diameter.

Since we found only one critical value, and the problem must have a solution (Why?), we can be sure that

the value  $x = k$  corresponds to a minimum of  $S$ . However, we can prove it analytically, by noting that

$$S'' = \pi \left( 1 + \frac{2k^3}{x^2} \right),$$

and this is positive when  $x = k$ .

*Method II.* Considering  $y$  as a function of  $x$ , we can write the derivative of  $S$  in the form

$$S' = \pi (x + y + xy').$$

By taking the derivative of (2), i.e.,

$$x^3y = k^3,$$

we obtain

$$x^2y' + 2xy = 0,$$

or

$$x(xy' + 2y) = 0.$$

Since  $x \neq 0$ , this gives  $xy' = -2y$  or  $y' = -\frac{2y}{x}$ ; substituting this value of  $y'$  in the expression for  $S'$ , we get

$$S' = \pi (x + y - 2y) = \pi (x - y).$$

Since for a minimum,  $S' = 0$ , this gives  $x = y$ , as before.

### PROBLEMS

1. Find two numbers whose sum is  $n$  such that their product is as large as possible.
2. Find two numbers whose sum is  $n$  such that the sum of their squares shall be a minimum.

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3. Find the largest rectangle with perimeter equal to  $2p$ .
4. A square piece of tin whose side is  $a$  has equal squares cut out at each corner. Find the side of the square cut out if the remainder forms a box of maximum capacity.
5. A rectangular piece of tin is  $7'' \times 15''$  and has equal squares cut out at each corner. Find the side of the square cut out if the remainder forms a box of maximum capacity.
6. Find the largest isosceles triangle that can be inscribed in a given circle.
7. A rectangular region including 60 sq. yd. is to be enclosed along a long straight wall any part of which can be used as one side of the enclosure. What lengths should the three new sides have to require the smallest amount of new material?
8. A tank has a square base and open top and holds 64 cu. ft. If the cost of the material of the sides is \$1. a sq. ft., and of the bottom \$2. a sq. ft., what are the dimensions of the box when the cost is a minimum?
9. In order that a package may go by parcel post the sum of its length and girth must not exceed 84 inches. What are the dimensions and the volume of the largest rectangular package with square ends that can be mailed?
10. What is the most economical shape of a cylindrical tin cup with circular bottom and open top which is to hold a half pint?
11. Find the area of the largest rectangle that can be inscribed in the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .
12. Find the altitude of the right circular cylinder of maximum volume inscribed in a sphere of radius  $r$ .

13. Find the altitude of the right circular cone of maximum volume inscribed in a sphere of radius  $r$ .
14. Find the coordinates of the point on  $2x + y = 16$ , such that the sum of the squares of its distances from  $(6 - 3)$  and  $(4, 5)$  is a minimum.
15. A rectangular building is to be erected on a triangular lot. Show that the maximum floor space available, if the thickness of the walls be neglected, is one-half the area of the lot.

**147. Velocity — Rates.** If a body, moving in a straight line, travels a distance  $s$  in the time  $t$ , then the average velocity  $v$  of the body is defined to be,

$$v = \frac{s}{t}.$$

For example, if an automobile goes 100 miles in 5 hours, its average velocity is 20 mi. per hour. This of course does not mean that it goes 20 mi. in each hour. In general, the average velocity in some hours will be greater than in others.

The velocity at any moment of time  $t$ , called the instantaneous velocity at the given instant, is found as follows. Let  $t + \Delta t$  be a later time and suppose in the time  $\Delta t$ , the body has moved the distance  $\Delta s$ . Then its average velocity during the time  $\Delta t$  is  $\frac{\Delta s}{\Delta t}$ . We now define the instantaneous velocity at the time  $t$  to be

$$v = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = D_t s.$$

Hence, the instantaneous velocity is the derivative of the distance  $s$  with respect to the time  $t$ . If  $t$  and  $D_t s$

are positive,  $s$  is increasing; if  $t$  is positive and  $D_t s$  is negative,  $s$  is decreasing.

In similar manner, if  $x$  is a function of  $t$  and if  $\Delta x$  and  $\Delta t$  are corresponding increments we define the instantaneous rate of change of  $x$  at the instant  $t$  to be,

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = D_t x.$$

The rate of change of the velocity is called the acceleration. If we denote it by  $a$ , we have

$$a = D_t v = D_t^2 s.$$

*Example 1.* Given  $s = 4t^2 - 3t + 2$ . Find the instantaneous velocity and acceleration at the time  $t = 1$ .

Solution:  $v = D_t s = 8t - 3$ ;

when  $t = 1$ ,  $v = D_t s = 8 - 3 = 5$ .

$$a = D_t v = 8.$$

*Example 2.* The distance  $s$  a body moves in a straight line in a time  $t$  measured from a fixed point on the line, is given by the formula  $s = t^2 - 6t + 3$ . Discuss the motion of the body.

Solution:  $s = t^2 - 6t + 3$ ,

$$v = D_t s = 2t - 6 = 2(t - 3).$$

When  $t > 3$ ,  $v$  is positive and  $s$  is increasing.

When  $t < 3$ ,  $v$  is negative and  $s$  is decreasing.

When  $t = 3$ ,  $v$  is zero and the body is at rest.

*Example 3.* Find the point on the parabola  $y^2 = 8x$

at which the abscissa and ordinate increase at the same rate.

Solution: By the conditions of the problem

$$D_t x = D_t y.$$

Differentiating the given equation we have

$$2 y D_t y = 8 D_t x.$$

$\therefore 2 y = 8$  or  $y = 4$ . Hence  $x = 2$  and the point is  $(2, 4)$ .

*Example 4.* A ladder 25 ft. long rests against a house. If the foot of the ladder is pulled along the ground away from the house at the rate of 3 ft. per minute, find how fast the top of the ladder is descending when it is 20 ft. from the ground.

Solution: As the ladder is being pulled away two distances change and are marked  $x$  and  $y$  in the adjacent figure. By the problem

$$D_t x = 3 \text{ when } y = 20;$$

we are asked to find  $D_t y$ .

Now, when  $y = 20$ ,  $20^2 + x^2 = 25^2$  or  $x = 15$ .

But 
$$x^2 + y^2 = 25^2.$$

$$\therefore 2 x D_t x + 2 y D_t y = 0,$$

or 
$$D_t y = -\frac{x}{y} D_t x.$$

$$D_t y = -\frac{15}{20} \cdot 3 = -\frac{9}{4} = -2.25 \text{ ft./min.}$$

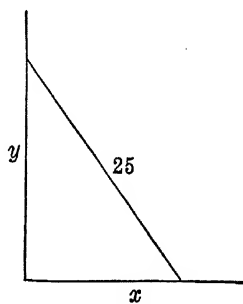


FIG. 169

## Exercises

In the following examples find the instantaneous velocity and the acceleration at the time indicated.

1.  $s = 7t^2 - 3t + 1$ ,  $t = 1$ .

2.  $s = 4t^3 + 3t - 1$ ,  $t = 2$ .

3.  $s = 5t^2 - 7t + 2$ ,  $t = 3$ .

4.  $s = \frac{3t-1}{t^2-1}$ ,  $t = 5$ .

Discuss the motion of the following bodies which move in a straight line according to the following laws.

5.  $s = t^2 - 4t + 3$ .

6.  $s = t^3 - 3t + 2$ .

7.  $s = 16t - 8t^2$ .

8.  $s = (t - 2)^2$ .

9. The height  $y$  in feet, of a ball thrown vertically upward after  $t$  seconds, is given by the formula

$$y = 96t - 16t^2.$$

- Find the velocity and acceleration at any instant.
  - Find the initial velocity (*i.e.*, when  $t = 0$ ).
  - Find the velocity and acceleration when  $t = 3$ .
  - Find the highest point which the ball reaches.
  - Find when the ball hits the ground.
10. The distance  $s$  an automobile goes on a straight road from its starting point is given by the formula  $s = \frac{1}{4}t^4 - 4t^3 + 16t^2$ .
- When will the automobile change its direction?
  - Describe the motion for the first 12 hours.
11. A particle moves on the parabola  $y^2 = 4x$ . If  $x$  increases

uniformly at the rate of 3 in. per second, find the rate at which  $y$  increases when  $x = 4$  in.

12. In the function  $y = 2x^3 - 2$ , find the value of  $x$  at the point where  $y$  increases 24 times as fast as  $x$ ?
13. Where in the first quadrant does the angle increase twice as fast as its sine?
14. Find the rate of change of the area of a square when its side is  $m$  ft. and is increasing  $n$  ft. per second.
15. For what value of  $x$  do the functions  $x^3 - 5x^2 + 17x$  and  $x^3 - 3x$  change at the same rate?
16. Given  $y = x^3 - 6x^2 + 3x + 5$ . Find the coordinates of the points at which the rate of change of the ordinate is equal to the rate of change of the slope of the tangent.
17. A man 6 ft. tall walks away from a lamp post 12 ft. high at the rate of 20 ft. a minute. Find how fast the further end of his shadow moves along the road. How fast does the length of the shadow increase?
18. When a stone is dropped into a pond, the radius of the circular wave increases 2 inches a second. How fast is the circumference increasing?
19. The volume of a soap bubble increases 2 cu. in. per sec. How fast is the radius increasing when it is 1 in.?
20. Water is flowing into a cylindrical tank at the rate of 5 cu. ft. per sec. If the radius of the tank is 4 ft., how fast is the water rising?
21. The cross-section of a tank is a rectangle 2 ft.  $\times$  3 ft. If water flows into the tank at the rate of 50 cu. ft. per minute, how fast is the water rising?
22. Two trains, one going east and the other south, start from

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- the same place and travel 40 and 60 mi. per hr. respectively. How fast are they separating at the end of 45 minutes?
23. A man on a dock 12 ft. above the water pulls in a rope attached to a boat at the rate of 3 ft. per minute. How fast is the boat approaching the dock when it is 16 ft. away?
  24. Water is flowing into a conical reservoir 24 in. deep, and 12 in. across the top, at the rate of 10 cu. in. per min. How fast is the surface rising when the water is 8 in. deep?
  25. Find the point on the ellipse  $16x^2 + 9y^2 = 400$  where  $y$  increases at the same rate  $x$  decreases.
  26. A kite is 100 ft. high and has 200 ft. of cord out. If the kite moves horizontally 3 mi. per hr. away from the boy who is flying it, how fast is the cord being let out?
  27. Sand being poured on the ground forms a pile in the shape of a right circular cone whose height is equal to the radius of the base. If the sand is falling at the rate of 5 cu. ft. per sec., how fast is the height of the pile increasing when it is 8 ft.?
  28. An airplane is 2640 ft. directly above an automobile. The airplane flies west at the rate of 100 mi. per hour, and the automobile goes east at the rate of 50 mi. per hour. How fast are they separating at the end of 6 minutes?

## ANSWERS

The answers to the odd exercises are omitted to give the student experience in checking his own work. Teachers desiring all the answers should communicate with the publishers.

Page 2. 2. Real, Equal,  $S = -1$ ,  $P = \frac{1}{4}$       4. Imaginary,  $S = \frac{1}{2}$ ,  $P = \frac{7}{6}$

6. Real, Equal,  $S = \frac{1}{2}$ ,  $P = \frac{1}{16}$       8.  $k = -9$ , 7

10.  $k = 3$ ,  $-\frac{1}{17}$       12.  $k = 1$ , 5

Page 3. 14.  $k = \pm 25$       16.  $k = \frac{p}{2}$

Page 4. 2.  $aq - bp$       4. 0

Page 5. 6. 50      8.  $k - 14 = 0$       10. 
$$\begin{vmatrix} (x^2 + y^2) & x & y & 1 \\ (x_1^2 + y_1^2) & x_1 & y_1 & 1 \\ (x_2^2 + y_2^2) & x_2 & y_2 & 1 \\ (x_3^2 + y_3^2) & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Page 8. 2.  $\sin \frac{\theta}{2} = \frac{1}{10}\sqrt{10}$ ,  $\cos \frac{\theta}{2} = \frac{3}{10}\sqrt{10}$

4.  $\tan(\alpha + \beta) = -\frac{33}{56}$ ,  $\tan(\alpha - \beta) = -\frac{63}{16}$

Page 9. 6.  $\sin 2\theta = \frac{4}{5}$ ,  $\cos 2\theta = \frac{3}{5}$ ,  $\sin \theta = \frac{1}{5}\sqrt{5}$ ,  $\cos \theta = \frac{2}{5}\sqrt{5}$

8.  $\tan \theta = 1$ ,  $\frac{1}{4}$

Page 10. 2.  $AD$       4.  $DA$       6. 0      8.  $AD = -2$ ,  $CB = 10$ ,  $AC = -3$   
10.  $DA = -4$ ,  $AB = 8$ ,  $BC = -1$

Page 11. 2. 0

Page 12. 4.  $|OM| = 5\sqrt{3}$ ,  $|ON| = 5$ .

Page 14. 2. a) -7, b) -1, c) 1, d) 2, e) 4, f) 7

In a) and b) the first point is to the right of the second.

In c), d), e), f) the first point is to the left of the second.

Page 16. 2. 0      4. (0, 0)      6. a) II      b) IV  
8. (-3, -4)      10. A (0, 0), B (10, 0), C (10, 3), D (0, 3)

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12. (-5, 0), (5, 0), (0,  $5\sqrt{3}$ ); (-5, 0), (5, 0), (0,  $-5\sqrt{3}$ )

14.  $2\sqrt{13}$

Page 19. 2. a)  $2\sqrt{a^2 + b^2}$ , b)  $\sqrt{(b-c)^2 + (a-b)^2}$

Page 20.

4. Isosceles                      8. (0, 10)                      10. 16, - 8  
 12. Isosceles                      14.  $x + y = 10$

Page 21. 16.  $5x^2 + 9y^2 = 45$ 

Page 24.

2.  $\left(-\frac{5}{2}, \frac{1}{2}\right), \left(\frac{5}{2}, \frac{3}{2}\right), (-2, 5)$       4.  $\left(\frac{46}{5}, \frac{32}{5}\right)$   
 6. (32, 27)                      8. (8, 4)                      10.  $\left(\frac{14}{3}, \frac{14}{3}\right)$

Page 25.

12.  $\left(\frac{22}{3}, \frac{4}{3}\right)$                       14. 10                      16. (4, 9). Three

Page 28.

2.  $-\frac{3}{5}$                       4.  $-\frac{6}{7}$                       6. no slope                      14. 6                      16. - 1                      18.  $\frac{50}{9}$

Page 30. 8.  $y = 3x - 5$ Page 31. 10.  $2x - y = 1$                       12. (16, 0), (-8, 12), (4, -4)

Page 39.

2.  $y = -3$                       4.  $x + 5y = 0$   
 6.  $(2b - a)x + (2a - b)y = 3a^2 + 3b^2$                       8.  $(x - a)^2 + (y - b)^2 = a$

Page 40. 10.  $x^2 + 4y + 4 = 0$ 12.  $x^2 + y^2 - 8x + 12 = 0$ 14.  $12x^2 + 12y^2 - 71x - 145y + 542 = 0$ 16.  $x^2 + y^2 + x - 9y + 16 = 0$ 18.  $28x^2 - 36y^2 = 63$ 20.  $44x^2 - 100y^2 = 275$ 

Page 44.

- |    | <i>x</i> -intercepts | <i>y</i> -intercepts |     | <i>x</i> -intercepts | <i>y</i> -intercepts |
|----|----------------------|----------------------|-----|----------------------|----------------------|
| 2. | - 1                  | $\pm 2$              | 8.  | none                 | 3                    |
| 4. | $\pm 2$              | none                 | 10. | 0                    | 0                    |
| 6. | 0                    | 0                    | 12. | none                 | $\pm 2$              |

Page 46.

- |    | <i>x</i> -axis | <i>y</i> -axis | origin |     | <i>x</i> -axis | <i>y</i> -axis | origin |
|----|----------------|----------------|--------|-----|----------------|----------------|--------|
| 2. | Yes            | No             | No     | 8.  | Yes            | No             | No     |
| 4. | Yes            | Yes            | Yes    | 10. | No             | No             | Yes    |
| 6. | No             | No             | Yes    | 12. | No             | No             | Yes    |

Page 47.

14. No                      No                      No                      16. No                      Yes                      No

Page 48. 2.  $y < 0$ 4.  $y > 0$ 6.  $x > 6, x < -6, y > 3, y < -3$ 

8. None

10.  $x > 0, x < -2, y > 1, y < -1$ 12.  $x > 0$

- Page 53. 2. (1, 1) 4. (1, 2) 6. (2, 4), (8, -8)  
 8. (4, 3),  $\left(\frac{8}{3}, \frac{1}{3}\right)$  10. (2, 1), (1, 2), (-1, -2), (-2, -1)  
 12. (0, 0), (0, 0),  $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{4}\right)$ ,  $\left(\frac{-\sqrt{3}}{2}, \frac{-\sqrt{3}}{4}\right)$

Page 58.

2.  $y = -2x - 5$  4.  $y = x + 4$  6.  $y = -x + 4$  8.  $by = ax + b^2$   
 10.  $x + 5y - 23 = 0$  12.  $x - y - 1 = 0$  14.  $\frac{3}{2}$

Page 59. 16.  $\frac{-a}{b}$ 18.  $\frac{-b}{a}$ 

22. a)  $4x - 3y - 30 = 0$  b)  $3x + 4y - 10 = 0$   
 24. a)  $Ax + By - (Aa + Bb) = 0$  b)  $-Bx + Ay + Ba - Ab = 0$

Page 60. 28. (1, 5), (5, 9), (3, -3)

30.  $8x + 11y - 70 = 0$ ,  $x + 3y + 1 = 0$ ;  $5x + 2y + 5 = 0$ .

Page 61.

2.  $x - y + 2 = 0$  4.  $7x - 3y + 4 = 0$  6.  $4x + y = 0$   
 8.  $x + y = a + b$  10.  $y - q = \frac{q-n}{p-m}(x-p)$

Page 62.

2.  $2x + y = -2$  4.  $2bx + 3ay = 6ab$   
 6.  $\frac{x}{8} + \frac{y}{16} = 1$  8.  $\frac{x}{25} + \frac{y}{25} = 1$   
 4.  $\frac{4}{8}$

Page 63. 2.  $y = 0$ 4.  $y = -1$ Page 64. 6.  $x = 6$ ,  $y = 7$ 

Page 65.

2.  $\sqrt{3}x + y = \pm 20$  4.  $x + y = -14$  6. No

2.  $\frac{x}{\frac{5}{3}} + \frac{y}{\frac{5}{4}} = 1$ ,  $y = -\frac{3}{4}x + \frac{5}{4}$ ,  $\frac{3}{5}x + \frac{4}{5}y = 1$

4.  $\frac{x}{-5} + \frac{y}{5} = 1$ ,  $y = \frac{1}{3}x + \frac{5}{3}$ ,  $\frac{-x}{\sqrt{10}} + \frac{3y}{\sqrt{10}} = \frac{5}{\sqrt{10}}$

6. Impossible,  $y = -\frac{p}{q}x$ ,  $-\frac{px}{\sqrt{p^2+q^2}} - \frac{qy}{\sqrt{p^2+q^2}} = 0$

Page 70.

$$8. 4x - 3y = 17, \quad 4x - 3y = -3 \quad 10. \frac{3x}{5} + \frac{4y}{5} = 5$$

Page 72.

$$2. 3x - 5y = -29 \quad 4. 3x - 2y = -2 \quad 6. 2x + 3y = -13$$

Page 73.

$$8. 4x - 3y = -24 \quad 10. 12x + 10y = 25$$

Page 74. 2. - 8

$$4. k = 5, \quad k = -1$$

Page 77.

$$2. \frac{1}{5} \quad 4. \frac{ab}{2a^2 + b^2} \quad 6. -\frac{1}{3} \quad 8. -1 \quad 10. \text{None. Lines perpendicular.}$$

Page 78.

$$12. x - 5y + 13 = 0 \quad 14. y + \frac{b}{2} = \frac{2ab}{a^2 - b^2} \left( x - \frac{a}{2} \right)$$

$$16. y + 5 = \frac{2 + \sqrt{3}}{1 - 2\sqrt{3}}(x - 8), \quad y + 5 = \frac{2 - \sqrt{3}}{1 + 2\sqrt{3}}(x - 8)$$

$$18. 0, \text{ no slope}$$

$$20. -1, -2, -3$$

$$2. a) \text{ Parallel } b) \text{ Perpendicular } c) \text{ None } d) \text{ None}$$

Page 79.

$$6. 1:3 \quad 8. c = 2d \quad 10. x - 5y - 16 = 0$$

$$12. 2x - 3y = 4 \quad 14. x + 2y = -4$$

Page 80.

$$16. 3x - 2y = -8, \quad 3x - 7y = -48, \quad y = 8. \text{ Intersect at } \left( \frac{8}{3}, 8 \right)$$

$$18. 3x + 2y = 34, \quad x - y = -8, \quad 4x + y = 26. \text{ Intersect at } \left( \frac{18}{5}, \frac{58}{5} \right)$$

$$20. 4x + y = 15, \quad 3x + 2y = 19, \quad x - y = -4. \text{ Intersect at } \left( \frac{11}{5}, \frac{31}{5} \right)$$

$$22. y = \sqrt{3}x + 6 - 3\sqrt{3}, \quad y = -\sqrt{3}x + 6 + 3\sqrt{3} \quad (3 - 2\sqrt{3}, 0), \\ (3 + 2\sqrt{3}, 0) \quad 24. -9$$

$$\text{Page 81. } 26. -\frac{7}{4}$$

Page 85.

$$2. -\frac{18}{5} \quad 4. \frac{23}{10}\sqrt{2}, \quad \frac{23}{41}\sqrt{41}, \quad \frac{23}{13}\sqrt{13}$$

$$6. \frac{7}{13}\sqrt{13}, \quad \frac{7}{41}\sqrt{41}, \quad \frac{7}{10}\sqrt{10} \quad 8. \frac{12}{5} \quad 10. \frac{3}{10}\sqrt{5}$$

Page 86.

$$14. \left(0, \frac{1}{2}\right), \left(0, \frac{11}{2}\right) \quad 16. 3x + 4y = 0, \quad 3x + 4y = 20$$

$$20. a) 7x + 7y - 19 = 0 \quad b) 17x + 17y + 13 = 0$$

Page 87.

$$22. a) 2x + 2y + 5 = 0, \quad x + 9 = 0, \quad x + 2y - 4 = 0, \quad \left(-9, \frac{13}{2}\right)$$

$$b) x - y + 10 = 0, \quad (\sqrt{2} + 1)y + x = 6\sqrt{2},$$

$$(\sqrt{2} + 1)x + y = -4\sqrt{2}, \quad (-6 + \sqrt{2}, 4 + \sqrt{2})$$

$$24. 27x + 1084y - 348 = 0, \quad 573x + 356y - 2532 = 0$$

Page 89. 2. 3 4. 0 6.  $\frac{45}{2}$ Page 92. 2.  $x + 3y = 0$ 

Page 93.

$$4. x - 2y - 1 = 0 \quad 6. 3x - y - 1 = 0 \quad 8. 5x - 10y - 27 = 0$$

$$10. x - 4y - 2 = 0 \quad 12. 43x + 13y = 599$$

Page 95.

$$2. \text{Line } x + 2y = 0 \quad 4. \text{Lines } y = x, y = 2x \quad 6. \text{Lines } x = -5, x = 1$$

$$8. \text{No locus} \quad 10. \text{Lines } x = 0, y = 0$$

Page 96. 12. Lines  $x - 2y - 1 = 0, x + y - 2 = 0$ 

Page 100.

$$2. (1, 1), (2, 3), \left(\frac{7}{2}, -\frac{3}{2}\right) \quad 4. (4, 2\sqrt{3}), (4, -2\sqrt{3}) \quad 6. x + 2y = 11$$

$$8. a) x - y - 2 = 0 \quad b) 4\sqrt{2} \quad c) 4\sqrt{2} \quad d) 16 \quad e) 16$$

Page 101.

$$f) x = 4, \quad x + 3y = 2, \quad 7x - 3y = 30 \quad g) \left(4, -\frac{2}{3}\right)$$

$$h) x + 5y = 2, \quad x - 3y = -6, \quad x + y = -2 \quad i) (-3, 1)$$

$$j) x + 5y = 0, \quad x + y = 6, \quad x - 3y = 12 \quad k) \left(\frac{15}{2}, -\frac{3}{2}\right)$$

$$14. \left(2, -\frac{1}{2}\right)$$

Page 102.

$$18. \left(\frac{74}{13}, \frac{90}{13}\right) \quad 20. (0, b) \quad 22. \left(\frac{28}{5}, \frac{56}{5}\right) \quad 24. y = x + 2$$

$$26. \text{Vertices } (5, 9), (9, 3). \text{ Equations of sides are } x + 5y - 50 = 0$$

$$x + 5y - 24 = 0, \quad 5x - y - 16 = 0, \quad 5x - y - 42 = 0$$

$$28. a) 7B + C = 0$$

Page 103.

$$\begin{array}{llll} 28. b) 3B + A = 0 & c) 5A + 2B = 0 & d) A = 0 & e) B = \\ f) 3A - 2B + C = 0 & 30. 2x - 5y + 29 = 0 & & \end{array}$$

Page 105.

$$\begin{array}{lll} 2. x^2 + y^2 = 36 & 4. x^2 + y^2 = 25 & 6. x^2 + y^2 - 2ax - 2ay = 0 \\ 8. x^2 + y^2 - 2x - 4y - 36 = 0 & 10. x^2 + y^2 - 6x - 8y + 9 = 0 & \\ 12. x^2 + y^2 - 2ax - 2ay + a^2 = 0 & & \\ 14. x^2 + y^2 - 6x - 4y + 9 = 0, & x^2 + y^2 - 6x + 4y + 9 = 0 & \\ 16. 169x^2 + 169y^2 + 676x - 1690y - 428 = 0 & & \\ 18. x^2 + y^2 - 8x - 8y + 16 = 0 & 20. x^2 + y^2 - 2x - 2y - 14 = 0 & \end{array}$$

Page 107.

$$\begin{array}{ll} 2. \text{Point } (-2, 1) & 4. \text{Circle, } C(-5, 12), R = 13 \\ 6. \text{Circle, } C(3, 0), R = 4 & 8. \text{Circle, } C\left(\frac{3}{2}, \frac{5}{2}\right), R = 3 \end{array}$$

Page 108. 10. No Locus

$$\begin{array}{ll} 12. x^2 - 2x + y^2 - 4y = 20 & 14. x^2 + 2x + y^2 - 8y = 0 \\ 16. x^2 - 6x + y^2 - 6y + 9 = 0, & x^2 - 6x + y^2 + 6y + 9 = 0 \\ 18. x^2 + y^2 - 2y = 9 & \end{array}$$

Page 111. 2.  $x^2 + y^2 + 2y - 1 = 0$ 

$$4. x^2 + y^2 + x - 4y + 1 = 0 \quad 6. 3(x^2 + y^2) + 17x - 16y + 25 = 0$$

Page 113. 2.  $2x + 3y + 1 = 0$ 

Page 114.

$$4. (1, 0), \left(-\frac{3}{5}, -\frac{24}{5}\right) \quad 6. (-3, 1), (1, -1)$$

8. (4, 1). Line is tangent to the circle

$$\begin{array}{ll} 10. a) 3x^2 + 3y^2 + 2x - 8y = 0 & b) 3x^2 + 3y^2 + x - 10y + 3 = 0 \\ c) x^2 + y^2 - 4x - 12y + 14 = 0 & \end{array}$$

Page 116. 2.  $2x + 3y + 13 = 0$ 

$$4. 3x + 4y = 25$$

$$6. 3x + y + 19 = 0 \quad 8. 3x + 4y = 20 \quad 10. ax + by = 2(a^2 + b^2)$$

Page 117.

$$2. C\left(-\frac{5}{4}, \frac{3}{4}\right), R = \frac{\sqrt{42}}{4} \quad 4. C\left(\frac{a}{2}, 0\right), R = \frac{1}{2}\sqrt{4k + a^2}$$

$$6. x^2 + y^2 - 5x - y = 0$$

$$8. x^2 - 6x + y^2 + 4y + 12 = 0$$

$$10. x^2 + y^2 - 21y + 74 = 0$$

$$12. x^2 - 3x + y^2 - y = 16$$

$$14. x^2 + y^2 - 2x - 2y + 1 = 0, \quad x^2 + y^2 - 10x - 10y + 25 = 0$$

$$16. \text{Points lie on } x^2 + y^2 - 6x + 4y - 12 = 0$$

Page 118.

$$18. x^2 + y^2 - 8x - 4y + 16 = 0, \quad x^2 + y^2 - 2x + 2y + 1 = 0$$

$$20. x^2 - 2x + y^2 - 2y + 1 = 0$$

Page 119. 24. Yes.  $x^2 + y^2 + 2x - 3y + 6 = 0$

Page 127. 2. V (0, 0); F (-1, 0);  $x = 1$ ; 4

Page 128.

4. V (0, 0); F (0, -1);  $y = 1$ ; 4

6. V (0, 0); F (-4, 0);  $x = 4$ ; 16

8. V (0, 0); F (0, -4);  $y = 4$ ; 16

10. V (0, 0); F  $\left(-\frac{7}{12}, 0\right)$ ;  $x = \frac{7}{12}$ ;  $\frac{7}{3}$

12. V (0, 0); F  $\left(0, -\frac{7}{12}\right)$ ;  $y = \frac{7}{12}$ ;  $\frac{7}{3}$

14.  $y^2 = -12x$

16.  $x^2 = -20y$

20.  $y^2 = 16x$

Page 129.

22.  $x^2 + 6y = 0$

24.  $x^2 + 16y + 64 = 0$

28.  $3x^2 + 4y = 0$

Page 130. 32. (1, 2); (1, 2)

34. (2, 4); (2, 4)

Page 133. 4.  $2b = 6\sqrt{3}$ ;  $2c = 6$

6.  $\frac{\sqrt{3}}{2}$

Page 136. 2.  $\frac{\sqrt{5}}{3}$

4.  $(\pm\sqrt{21}, 0)$ ;  $e \frac{\sqrt{21}}{5}$

6. (a) F  $(\pm\sqrt{5}, 0)$ ; V  $(\pm 3, 0)$  (b) F  $(0, \pm\sqrt{5})$ ; V  $(0, \pm 3)$

(c) F  $(0, \pm\sqrt{21})$ ; V  $(0, \pm 5)$  (d) F  $(\pm\sqrt{21}, 0)$ ; V  $(\pm 5, 0)$

Page 141.

2. a) 5, 3

b)  $(\pm 4, 0)$ ,  $(\pm 5, 0)$

c)  $\frac{4}{5}$

d)  $c = \pm \frac{25}{4}$

e)  $\frac{18}{5}$ ;  $\left(4, \pm \frac{9}{5}\right)$ ;  $\left(-4, \pm \frac{9}{5}\right)$

4. a) 4, 3

b)  $(\pm\sqrt{7}, 0)$ ;  $(\pm 4, 0)$

c)  $\frac{\sqrt{7}}{4}$

d)  $x = \pm \frac{16}{7}\sqrt{7}$

e)  $\frac{9}{2}$ ;  $\left(\sqrt{7}, \pm \frac{9}{4}\right)$ ;  $\left(-\sqrt{7}, \pm \frac{9}{4}\right)$

6. a)  $\frac{\sqrt{115}}{5}$   $\sqrt{69}$

b)  $\left(0, \pm \frac{\sqrt{2415}}{30}\right)$   $\left(0, \pm \frac{\sqrt{115}}{5}\right)$

c)  $\frac{\sqrt{21}}{6}$

d)  $y = \pm \frac{2}{35}\sqrt{2415}$

e)  $\frac{\sqrt{115}}{6}$ ,  $\left(\pm \frac{\sqrt{115}}{12}, \frac{\sqrt{2415}}{30}\right)$ ,  $\left(\pm \frac{\sqrt{115}}{12}, -\frac{\sqrt{2415}}{30}\right)$

8. a) 5, 3      b)  $(0, \pm 4), (0, \pm 5)$       c)  $e = \frac{4}{5}$

d)  $y = \frac{25}{4}$       e)  $\frac{18}{5}, \left(\pm \frac{9}{5}, 4\right), \left(\pm \frac{9}{5}, -4\right)$

10. a) 4, 3      b)  $(0, \pm \sqrt{7}), (0, \pm 4)$       c)  $\frac{\sqrt{7}}{4}$

d)  $y = \pm \frac{16}{7} \sqrt{7}$       e)  $\frac{9}{2}, \left(\pm \frac{9}{4}, \sqrt{7}\right); \left(\pm \frac{9}{4}, -\sqrt{7}\right)$

12. a)  $\frac{9}{5}, \frac{6}{5}$       b)  $\left(\pm \frac{3\sqrt{5}}{5}, 0\right), \left(\pm \frac{9}{5}, 0\right)$       c)  $\frac{\sqrt{5}}{3}$

d)  $x = \pm \frac{27}{25} \sqrt{5}$       e)  $\frac{8}{5}, \left(\frac{3\sqrt{5}}{5}, \pm \frac{4}{5}\right), \left(-\frac{3\sqrt{5}}{5}, \pm \frac{4}{5}\right)$

14.  $x^2 + 5y^2 = 9$

16.  $16x^2 + 25y^2 = 400$

18.  $20x^2 + 36y^2 = 405$

20.  $3x^2 + 4y^2 = 48$

Page 142.

22.  $81x^2 + 45y^2 = 500$

24.  $(0, 4); (0, -4)$

26. Circle; 0; 0; 2a; no value

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4.  $\frac{10}{3}, \frac{5\sqrt{5}}{3}$

6.  $e = \frac{3}{2}; b = \sqrt{5}$

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2.  $\frac{5}{3}$

4.  $(\pm \sqrt{29}, 0); e = \frac{\sqrt{29}}{5}$

Page 157.

2. a) 3, 4      b)  $F(\pm 5, 0); V(\pm 3, 0)$       c)  $\frac{5}{3}$       d)  $x = \pm \frac{9}{5}$

e)  $4x - 3y = 0, 4x + 3y = 0$       f)  $\frac{32}{3}, \left(5, \frac{16}{3}\right), \left(5, -\frac{16}{3}\right)$

4. a) 2, 2      b)  $F(\pm 2\sqrt{2}, 0); V(\pm 2, 0)$       c)  $\sqrt{2}$       d)  $x = \pm \sqrt{2}$

e)  $x - y = 0, x + y = 0$       f) 4;  $(2\sqrt{2}, \pm 2); (-2\sqrt{2}, \pm 2)$

6. a)  $\frac{5}{2}, \frac{5}{3}$       b)  $\left(\pm \frac{5}{6} \sqrt{13}, 0\right), \left(\pm \frac{5}{2}, 0\right)$       c)  $\frac{\sqrt{13}}{3}$

d)  $x = \pm \frac{15}{26} \sqrt{13}$       e)  $2x - 3y = 0, 2x + 3y = 0$

f)  $\frac{20}{9}, \left(\frac{5}{6} \sqrt{13} \pm \frac{10}{9}\right), \left(-\frac{5}{6} \sqrt{13}, \pm \frac{10}{9}\right)$

$$8. a) 3, 4 \quad b) F(0, \pm 5); \quad V(0, \pm 3) \quad c) \frac{5}{3}$$

$$d) y = \pm \frac{y}{5} \quad e) 3x - 4y = 0, 3x + 4y = 0$$

$$f) \frac{32}{3}, \left( \pm \frac{16}{3}, 5 \right); \quad \left( \pm \frac{16}{3}, -5 \right)$$

$$10. a) 5, 3 \quad b) F(0, \pm \sqrt{34}); \quad V(0, \pm 5) \quad c) \sqrt{34}$$

$$d) y = \pm \frac{25}{34} \sqrt{34} \quad e) 5x - 3y = 0, 5x + 3y = 0$$

$$f) \frac{18}{5}; \quad \left( \pm \frac{9}{5}, \sqrt{34} \right); \quad \left( \pm \frac{9}{5}, -\sqrt{34} \right)$$

$$12. a) \frac{5}{2}, \frac{5}{3} \quad b) \left( 0, \pm \frac{5}{6} \sqrt{13} \right), \quad \left( 0, \pm \frac{5}{2} \right) \quad c) \frac{\sqrt{13}}{3}$$

$$d) y = \pm \frac{15}{26} \sqrt{13} \quad e) 2y - 3x = 0, 2y + 3x = 0$$

$$f) \frac{20}{9}; \quad \left( \pm \frac{10}{9}, \frac{5}{6} \sqrt{13} \right); \quad \left( \pm \frac{10}{9}, -\frac{5}{6} \sqrt{13} \right)$$

$$14. 16x^2 - 9y^2 = 144$$

$$16. 16x^2 - 9y^2 = 256$$

$$18. 5x^2 - 20y^2 = 36$$

$$20. 7y^2 - 9x^2 = 175$$

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$$22. 144x^2 - 81y^2 = 1600 \quad 24. 4x^2 - y^2 = 20 \quad 26. 5x^2 - 45y^2 = 72$$

$$\text{Page 161. } 2. y^2 = 4(x^2 + 1) \quad 4. xy = k$$

$$6. xy = 8; \quad (2\sqrt{2}, 2\sqrt{2}), \quad (-2\sqrt{2}, -2\sqrt{2}) \quad 8. \sqrt{2}$$

$$\text{Page 163. } 6. x^2 - y^2 = 16$$

Page 168.

$$2. 2x_1 - 3 \quad 4. \frac{1}{y_1} \quad 6. -\frac{4x_1}{3}$$

$$8. 6x - y - 2 = 0 \quad 10. y + 3x = 21 \quad 12. 3x - y = 0$$

Page 170.

$$2. x - y + 2 = 0 \quad 4. 3x - 2y = 5$$

$$6. 2Ax_1x - B(y + y_1) = 0 \quad 8. x + 2y = 8$$

Page 171.

$$2. x - 3y - 9 = 0 \quad 4. b^2x_1(y - y_1) = a^2y_1(x - x_1)$$

Page 175.

$$2. y = 2x + 1 \quad 4. x - 2y + 16 = 0 \quad 6. x - y + 1 = 0, \quad x + y + 1 = 0$$

$$8. x + y + 4 = 0, \quad 2x + y + 2 = 0 \quad 10. x - 4y + 8 = 0, \quad x + 2y + 2 = 0$$

Page 176.

$$12. y = -5x \pm 4 \quad 14. x + y + 1 = 0 \quad 16. 9x - 2y = 5, \quad 9x + 2y = 5$$

Page 181. 2.  $y = -x \pm 2$ Page 182. 6.  $4x - y = 5, x + 4y + 3 = 0$ 

$$8. \frac{169}{40} \quad 10. \left. \begin{array}{l} 25x + 6y = 137 \\ 6x - 25y = -20 \end{array} \right\} \quad \left. \begin{array}{l} 25x - 6y + 137 = 0 \\ 6x + 25y = 20 \end{array} \right\}$$

$$12. 9x + 15y = 125 \quad 16. \sqrt{7}x + 4y = 16, \quad 16x - 4\sqrt{7}y = 7\sqrt{7}$$

Page 183. 18.  $x - y - 5 = 0, \quad x + 4y - 10 = 0$ Page 190. 2.  $x^2 + y^2 = 19$ 

Page 191.

$$4. y^2 - 6y' - 8x' - 31 = 0 \quad 6. V(1, 4), F\left(\frac{3}{2}, 4\right), x = \frac{1}{2}$$

$$8. V(-3, -5), F\left(-3, -\frac{15}{4}\right), y = \frac{-25}{4}$$

$$10. V(-1, -2), F(0, -2), x = -2$$

$$12. C(1, 1), V_1(4, 1), V_2(-2, 1), F_1(1 + \sqrt{5}, 1), F_2(1 - \sqrt{5}, 1), \\ x = 1 \pm \frac{9}{5}\sqrt{5}$$

$$14. C(1, -2), V_1(1, 3), V_2(1, -7), F_1(1, 2), F_2(1, -6), \\ y = \frac{17}{4}, y = \frac{-33}{4}$$

$$16. C(1, -2), V_1(1, 0), V_2(1, -4), F_1(1, -2 + \sqrt{3}), \\ F_2(1, -2 - \sqrt{3}), y = -2 \pm \frac{4}{3}\sqrt{3}$$

$$18. C(-2, 2), V_1(-2, 5), V_2(-2, -1), F_1(-2, 2 + \sqrt{5}), \\ F_2(-2, 2 - \sqrt{5}), y = 2 \pm \frac{9}{5}\sqrt{5}$$

$$20. C(-3, 2), V_1\left(-3 + \frac{3}{2}\sqrt{2}, 2\right), V_2\left(-3 - \frac{3}{2}\sqrt{2}, 2\right), \\ F_1\left(-3 + \frac{3}{2}\sqrt{3}, 2\right), F_2\left(-3 - \frac{3}{2}\sqrt{3}, 2\right), x = -3 \pm \sqrt{3}, \\ \sqrt{2}x - 2y + 3\sqrt{2} + 4 = 0, \quad \sqrt{2}x + 2y + 3\sqrt{2} - 4 = 0$$

Page 193. 2.  $x^2 + y^2 = 25$ Page 194. 4.  $x^2 - y^2 = 2k$ 

$$6. x^2 - y^2 = 8 \quad 8. 17x^2 - 17y^2 = 40$$

Page 197. 2.  $\theta = \arctan 2, 25x^2 - 72\sqrt{5}x - 36\sqrt{5}y = 0$ 

$$4. \theta = \arctan \frac{3}{2}, 7x^2 - 6y^2 = 42$$

$$6. \theta = \arctan \frac{\sqrt{3}}{3} \quad 2y^2 - 3x' + \sqrt{3}y' = 0$$

$$8. \theta = 45^\circ, \quad 6x^2 - 2y^2 + \sqrt{2}x' + 13\sqrt{2}y' = 16$$

Page 198. 2.  $x^2 - xy - 2y^2 - x + 2y = 0$

$$4. 48x^2 - 11xy - 17y^2 - 129x + 24y + 81 = 0$$

Page 211. 4.  $y = 4x, \quad 16y = -9x$

Page 212.

$$6. y = -6 \quad 10. 4x + 9y = 22 \quad 12. 5x - 2y - 11 = 0$$

Page 217.

$$2. a) \left(3, -\frac{1}{2}\right) \quad b) (-4, -10) \quad c) \left(\frac{9}{5}, \frac{21}{25}\right)$$

Page 220.

$$\begin{aligned} 2. a) & (4, 60^\circ), \quad (-4, 240^\circ), \quad (-4, -120^\circ), \quad (4, -300^\circ) \\ b) & (2, 30^\circ), \quad (-2, 210^\circ), \quad (-2, -150^\circ), \quad (2, -330^\circ) \\ c) & (-2, 45^\circ), \quad (2, 225^\circ), \quad (2, -135^\circ), \quad (-2, -315^\circ) \\ d) & (3, -150^\circ), \quad (-3, 30^\circ), \quad (3, 210^\circ), \quad (-3, -330^\circ) \\ e) & (-2, -100^\circ), \quad (2, 80^\circ), \quad (2, -280^\circ), \quad (-2, 260^\circ) \\ f) & (-2, -180^\circ), \quad (-2, 180^\circ), \quad (2, 0^\circ) \\ g) & (-3, 120^\circ), \quad (3, -60^\circ), \quad (-3, -240^\circ), \quad (3, 300^\circ) \\ h) & (-2, 270^\circ), \quad (2, 90^\circ), \quad (2, -270^\circ), \quad (-2, -90^\circ) \\ i) & (-3, 180^\circ), \quad (-3, -180^\circ), \quad (3, 0^\circ) \\ j) & (3, 240^\circ), \quad (3, -120^\circ), \quad (-3, 60^\circ), \quad (-3, -300^\circ) \\ k) & (0, \text{any angle}) \\ l) & (5, 330^\circ), \quad (5, -30^\circ), \quad (-5, 150^\circ), \quad (-5, -210^\circ) \end{aligned}$$

Page 221. 8.  $3\sqrt{3}$

Page 227.

$$\begin{aligned} 2. y &= 5 & 4. x^2 - y^2 &= 4 & 6. x^2 &= 2ay \\ 8. \frac{y}{x} &= \tan \frac{1}{\sqrt{x^2 + y^2}} & 10. (x^2 + y^2 + x)^2 &= x^2 + y^2 \end{aligned}$$

Page 228.

$$\begin{aligned} 12. r(\cos \theta + \sin \theta) &= 10 & 14. r^2 \sin 2\theta &= 20 & 16. r &= 2 \cos \theta \pm 1 \\ 18. r &= 2 \tan \theta \sin \theta & 20. r \cos(\theta - \alpha) &= p \end{aligned}$$

Page 231. 2.  $r = \frac{ep}{1 - e \cos \theta}$   $e > 1$

Page 237. 6.  $r = 4d \cos \theta$

### Miscellaneous Exercises

$$4. y^2 = 4ax + 4a^2$$

Page 238.

$$6. \frac{1}{2} r_1 r_2 \sin (\alpha_2 - \alpha_1)$$

$$8. r = \frac{6}{3 - 2 \cos \theta}$$

Page 239. 10.  $r = a \sec \theta \pm b$ Page 243. 10.  $(pt_1t_2, p(t_1 + t_2))$ 

$$Page\ 255. \quad 2. d = 7\sqrt{2}, \frac{4}{7}\sqrt{2}, \frac{3}{14}\sqrt{2}, \frac{-5}{14}\sqrt{2}$$

$$Page\ 256. \quad 4. d = 7; \quad \frac{6}{7}, \frac{3}{7}, \frac{2}{7}$$

$$6. \left(\frac{9}{5}, \frac{4}{5}, 1\right), (-1, 5, -6) \quad 8. \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}$$

10. Sphere, center at origin, radius 5

$$Page\ 260. \quad 4. \lambda: \mu: \nu = 1: -2: 1 \quad 6. \frac{\sqrt{14}}{z}$$

Page 262

$$2. \left(\sqrt{6}, 45^\circ, \tan^{-1} \frac{\sqrt{2}}{2}\right), \left(\sqrt{3}, 45^\circ, \tan^{-1} \sqrt{2}\right)$$

$$\left(3, \tan^{-1} 2, \tan^{-1} \frac{\sqrt{5}}{2}\right), \left(-2\sqrt{3}, 45^\circ, \tan^{-1} \sqrt{2}\right)$$

$$4. (\sqrt{2}, 45^\circ, 2), (\sqrt{2}, 45^\circ, -4), (-2\sqrt{2}, 135^\circ, 2)$$

$$6. (7,052,935, 2565), 3017 \text{ miles}$$

Page 270. 2.  $Ax + By + Cz = 0$ 

$$4. a) 3, -4, 12 \quad b) -4, 4, -4$$

$$c) 0, 0, \text{ no intercept} \quad d) \frac{-D}{A}, \frac{-D}{B}, \frac{-D}{C}$$

$$6. a) 2x - y + z = 3 \quad b) 2x - y - z = 0 \quad c) x + z = 4$$

Page 271.

$$8. a) 8x + 9y + 12z = 29 \quad b) 13x - 3y + 11z = 14 \quad c) 2x + 8y + 5z = 27$$

$$12. a) \frac{3}{14}\sqrt{7} \quad b) \frac{5}{186}\sqrt{93} \quad c) \frac{25}{38}$$

Page 272.

$$14. a) \frac{3}{26}\sqrt{26} \quad b) \frac{8}{19}\sqrt{38} \quad c) \frac{1}{3}\sqrt{6}$$

Page 276.

$$4. a) 2x + 6y - z - 7 = 0 \quad b) 9x + 27y - 167z + 122 = 0$$

Page 279.

$$2. a) \frac{x-3}{1} - \frac{y-4}{-1} - \frac{z-5}{6} \quad c) x = -2 + 6t, \quad y = 3 + t, \quad z = 2$$

$$b) \frac{x-4}{2} = \frac{y-6}{7} = \frac{z-5}{1}$$

$$4. a) \frac{x-2}{4} = \frac{y+1}{-3} = \frac{z-3}{2}$$

$$b) \frac{x-1}{3} = \frac{y}{-2} = \frac{z-2}{-1}$$

$$c) \frac{x-2}{5} = \frac{y+1}{2} = \frac{z}{-7}$$

$$\text{Page 280. } 6. \frac{x-7}{1} = \frac{y+2}{-6} = \frac{z-4}{22}$$

$$2. a) 20x + 5y + 4z = 20 \quad b) 2x - y + 3z = 6 \quad c) x - 4y - 4z = 4$$

Page 281.

$$4. 4x + 3y - 2z = 1 \quad 6. \frac{2}{5}\sqrt{2}, \frac{1}{2}\sqrt{2}, \frac{-3}{10}\sqrt{2}; d = \frac{2}{5}\sqrt{2}$$

8. No

$$10. 6x + y - 4z = -12$$

Page 282.

$$12. a) \frac{x}{\frac{4}{7}} + \frac{y}{\frac{2}{3}} + \frac{z}{-\frac{4}{3}} = 1; \frac{7}{\sqrt{86}}x + \frac{6}{\sqrt{86}}y - \frac{1}{\sqrt{86}}z = \frac{4}{\sqrt{86}}$$

$$b) \frac{x}{\frac{1}{5}} + \frac{y}{-\frac{4}{3}} + \frac{z}{\frac{8}{3}} = 1; \frac{8}{\sqrt{173}}x - \frac{10}{\sqrt{173}}y + \frac{3}{\sqrt{173}}z = \frac{8}{\sqrt{173}}$$

$$14. \cos \theta = \frac{\sqrt{231}}{77}$$

$$18. \cos \theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\text{Page 283. } 20. 6x - y - 3z + 4 = 0$$

22. Yes

$$\text{Page 284. } 26. k = 0, -3$$

Page 288.

2. a) No locus

e) Point  $(-1, -2, -3)$

$$b) C(-1, -2, -3), R = \sqrt{13}$$

$$f) C\left(-\frac{5}{6}, 1, -\frac{7}{6}\right), R = \frac{\sqrt{110}}{6}$$

$$c) C(-1, 0, 0), R = 1$$

g) Not a sphere

$$d) C(-1, -1, 0), R = \sqrt{2}$$

$$4. (x-a)^2 + (y-a)^2 + (z-a)^2 = a^2$$

$$(x-a)^2 + (y-a)^2 + (z+a)^2 = a^2$$

$$(x-a)^2 + (y+a)^2 + (z-a)^2 = a^2$$

$$(x+a)^2 + (y-a)^2 + (z-a)^2 = a^2$$

$$(x-a)^2 + (y+a)^2 + (z+a)^2 = a^2$$

$$(x+a)^2 + (y-a)^2 + (z+a)^2 = a^2$$

$$(x+a)^2 + (y+a)^2 + (z-a)^2 = a^2$$

$$(x+a)^2 + (y+a)^2 + (z+a)^2 = a^2$$

$$\text{Page 294. } 2. 4x^2 + 4z^2 + 9y^2 = 36$$

$$4. 4x^2 + 4y^2 + 9z^2 = 36$$

$$6. x^2 + z^2 = 4y$$

$$8. x^2 - y^2 + z^2 + 16 = 0$$

$$10. x^2 + y^2 + z^2 = 4$$

$$12. y^2 + z^2 = 9x^2$$

14.  $(0, 0, 0)$ . No locus

Page 301.

2. Elliptic hyperboloid of one sheet,  $C(0, 0, 0)$ 

4. Hyperbolic paraboloid      6. Two planes      8. No locus

Page 302

10. Families of lines are:

$$\left\{ \begin{array}{l} \frac{x}{3} + \frac{y}{4} = t \left( 1 + \frac{z}{5} \right) \\ t \left( \frac{x}{3} - \frac{y}{4} \right) = 1 - \frac{z}{5} \end{array} \right\} \quad \left\{ \begin{array}{l} \frac{x}{3} + \frac{y}{4} = t \left( 1 - \frac{z}{5} \right) \\ t \left( \frac{x}{3} - \frac{y}{4} \right) = 1 + \frac{z}{5} \end{array} \right\}$$

Page 305.

$$2. 1, \frac{1}{4} \quad 2. \frac{1}{x^2 + 3x} - \frac{1}{x} \quad 4. y = -\frac{A}{B}x - \frac{C}{B} \quad 6. y = \pm \frac{b}{a} \sqrt{x^2 - a^2}$$

Page 307.    2. 2      4. 2      6. -4      8. No limit exists.

$$Page 310. \quad 2. 2x + 1 \quad 4. 3x^2 \quad 6. \frac{-2}{x^2}$$

Page 311.

$$8. 1 - \frac{1}{x^2} \quad 10. \frac{-x}{y} \quad 12. \frac{2x}{y} \quad 14. -1$$

Page 314.    2.  $2x - 2$ ; 4

$$4. 4x^2 - 8x; 16$$

$$6. x = 1, -1; -3; (1, -2), (-1, 2), (0, 0)$$

$$8. a) y - 8x + 16 = 0, \quad x + 8y = 2$$

$$b) y - 4x + 9 = 0, \quad 4y + x - 15 = 0$$

$$c) y - 3x = 0; \quad 3y + x - 10 = 0$$

$$d) y - 16x + 31 = 0; \quad 16y + x - 18 = 0$$

$$e) y + 15x - 9 = 0; \quad 15y - x + 91 = 0$$

Page 318.

$$2. 5x^4 + 3x^2 - 2x \quad 4. 7x^4 - 8x^3 + 11x^2 - 8x + 4$$

$$6. \frac{2}{(x+1)^2} \quad 8. \frac{-3}{x^4}$$

Page 321.

$$2. 6x(x^2 - 1)^2 \quad 4. (30x - 10)(3x^2 - 2x - 7)^4$$

$$6. \frac{2}{y} \quad 8. \frac{x}{4y}$$

Page 322.

$$10. \frac{7 + 2y - 2x}{2y - 2x} \quad 12. \frac{1}{3x^{3/2}} \quad 14. \frac{x}{\sqrt{x^2 + 7}}$$

$$16. \frac{3x - 3}{2(3x^2 - 6x - 2)^{3/4}} \quad 18. \frac{(4x^2 + x + 6)\sqrt{x^2 + 3}}{(x^2 + 3)} \quad 20. \frac{-15}{y(5x - 2)^2}$$

Page 323.

$$2. 3x - 2y - 1 = 0, \quad 2x + 3y - 5 = 0; \quad 3x - y - 4 = 0$$

$$x + 3y - 28 = 0$$

$$4. 8x - y = 12, \quad x + 8y = 34$$

$$6. x - 12y + 11 = 0, \quad 12x + y - 13 = 0$$

Page 325. 2. Positive values increasing, negative decreasing

Page 326. 4. Increases  $x < 0$ ,  $x > 2$ ; Decreases  $0 < x < 2$ 

$$6. \text{Increases } -\frac{1}{2} < x < 0, \quad x > 2; \text{Decreases } x < -\frac{1}{2}, \quad 0 < x < 2$$

Page 327.

$$2. \text{Increases } x > 0, \quad \text{Decreases } x < 0. \quad \text{Turning point } (0, 5)$$

$$4. \text{Increases } x < -3, \quad x > 3; \quad \text{Decreases } -3 < x < 3$$

$$\text{Turning points } (-3, 60); \quad (3, -48)$$

$$6. \text{Increases } -\frac{1}{2} < x < 0; \quad x > 2; \text{Decreases } x < -\frac{1}{2}, \quad 0 < x < 2$$

$$\text{Turning points } \left(-\frac{1}{2}, -\frac{115}{16}\right), \quad (0, -7), \quad (2, -15)$$

$$8. \text{Decreases for all values of } x. \quad \text{No turning points}$$

$$\text{Page 337. } 2. \frac{n}{2}, \quad \frac{n}{2}$$

$$\text{Page 338. } 4. \frac{a}{6}$$

$$6. \text{Alt.} = \frac{1}{2} \text{base} \times \sqrt{3}$$

$$8. 4 \times 4 \times 4$$

$$10. \text{Radius of base} = \text{height}$$

$$12. \text{Alt.} = \frac{2r}{3} \sqrt{3}$$

Page 339. 14. (7, 2)

Page 342. 2.  $v = 51$ ,  $a = 48$ 

$$4. v = -\frac{17}{144}, \quad a = \frac{29}{576}$$

$$6. t < -1, \quad v \text{ positive, } s \text{ increases}$$

$$t = -1, \quad \text{body at rest}$$

$$-1 < t < 1, \quad v \text{ negative, } s \text{ decreases}$$

$$t = 1, \quad \text{body at rest}$$

$$t > 1, \quad v \text{ positive, } s \text{ increases}$$

$$8. t < 2, \quad v \text{ negative, } s \text{ decreases}$$

$$t = 2, \quad \text{body at rest}$$

$$t > 2, \quad v \text{ positive, } s \text{ increases}$$

$$10. a) t = 4, \quad t = 8 \quad b) \text{right for 4 hrs., left for 4 hrs., right 4 hrs}$$

*Page 343.*

12.  $x = 2, -2$

18.  $4\pi$  in./sec.

24.  $\frac{5}{2\pi}$  in./min.

14.  $2mn$  sq. ft.

20.  $\frac{5}{16\pi}$  ft./sec.

26.  $\frac{3}{2}\sqrt{3}$  mi./hr.

16.  $(1, 3), (5, -5)$

22.  $20\sqrt{13}$  mi./hr.

28.  $\frac{4500}{901}\sqrt{901}$

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